Nonabelian mixed Hodge structures

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Abstract— We propose a definition of "nonabelian mixed Hodge structure" together with a construction associating to a smooth projective variety X and to a nonabelian mixed Hodge structure V, the "nonabelian cohomology of X with coefficients in V" which is a (pre-)nonabelian mixed Hodge structure denoted $H = Hom(X_M, V)$. We describe the basic definitions and then give some conjectures saying what is supposed to happen. At the end we compute an example: the case where V has underlying homotopy type the complexified 2-sphere, and mixed Hodge structure coming from its identification with $\mathbf{P}^1$. For this example we show that $Hom(X_M, V)$ is a namhs for any smooth projective variety X.

Introduction

The nonabelian aspect of homotopy theory has been present in mixed Hodge theory from very early on. This started with the work of Deligne-Griffiths-Morgan-Sullivan [11] on rational homotopy theory, and then Morgan [45] put a mixed Hodge structure on the differential graded algebra which calculates the homotopy groups of a simply connected complex variety (or the nilpotent completion of the fundamental group of a non-simply connected one). This construction was later improved by Hain [26] [27] who in particular obtained a much better functoriality.

Many applications of the Morgan-Hain construction have been given, for example in Torelli-type theorems [9], [10], the use of the strictness property [40], in the study of algebraic cycles [24], [70], for the Shafarevich conjecture for nilpotent fundamental groups [37] [42], etc. There have recently been new attempts to better understand what is going on, see [3], [49], and to simplify the relatively abstract nature of Hain's construction, see [42], [34].

In an important development, Hain obtained a mixed Hodge structure on the *relative Malcev completions* of the fundamental group at any representation underlying a variation of Hodge structure [28]. This significantly extends the "amount" of the fundamental group that is seen by mixed Hodge theory.

The variation of the mixed Hodge structures on homotopy groups, when the underlying variety varies in a family, has been studied by Navarro-Aznar [46] [47] and Wojtkowiac [72].

There are two basic gaps in the classical mixed Hodge homotopy theory. The first is that one should really treat a "higher functoriality": the set of homotopy n-types forms an $n + 1$-category, so one would like to have an $n+1$-category of "homotopy n-types with mixed Hodge structure". The fact that we will then have an $n + 1$-functor from the 1-category of varieties, to this $n + 1$-category, contains subtle additional "homotopy coherence" data.

The second gap is that up until fairly recently, very little has been done on the problem of the higher homotopy theory of non-simply connected varieties. A first look at this problem may be considered to come from the work of Green-Lazarsfeld [22] on cohomology jump loci, although they don't talk about mixed Hodge structures. Arapura has looked at the Green-Lazarsfeld results using mixed Hodge structures [1]. Kapranov [35] and Toen [69] also have methods which go in the direction of looking at the higher homotopy

theory of non-simply connected varieties, but these haven't been fully developed yet.

The third author of the present paper has proposed, as a remedy to the above problems, to look at "nonabelian cohomology". One chooses a "coefficient object" which is an n-stack T, and looks at the *nonabelian cohomology* $H = \underline{Hom}(X_B, T)$. The coefficient object encodes data which is much the same as that of a complex differential graded algebra, however on the level of the fundamental group of T we have $\pi_1(T) = G$ an affine group scheme. The cohomology stack H now lies over the moduli 1-stack $\mathcal{M}(X_B, G) = \underline{Hom}(X_B, BG)$ of representations of $\pi_1(X^{\text{top}})$ in G, and H encodes higher homotopy data relative to the universal family of representations of $\pi_1(X^{\text{top}})$. In this way we get an approach towards the problem of higher homotopy in the presence of a fundamental group. The construction $X \mapsto \underline{Hom}(X_B, T)$ is automatically $n + 1$-functorial in X (in T too), so this addresses the issue of "higher functoriality". Furthermore, this higher functoriality shows up in concrete ways, for example in the case of a family of varieties one obtains a secondary Kodaira-Spencer map [65] which detects a "higher variation" of Hodge structure even when all of the classical variations of (mixed) Hodge structure associated to the family are constant.

One of the major problems in this nonabelian cohomology approach has been the lack of a notion of weight filtration, and consequently the lack of a notion of mixed Hodge structure. The present paper aims to start to close this gap. Briefly, what we do is to propose a definition of nonabelian mixed Hodge structure and in particular, the notion of "weight filtration" that goes into this. We state a number of conjectures about the notion that we define, but only go a short way toward justifying these conjectures by explicitly treating a particular example in Part III.

Our search for a notion of nonabelian mixed Hodge structure is motivated by the possibility of obtaining applications similar to what has already been done with mixed Hodge structures on homotopy groups. One of the main things would be to obtain restrictions on the homotopy types of smooth projective varieties (or even general quasiprojective varieties). This might include generalized "quadratic singularity" statements, for nonabelian cohomology stacks. Other possibilities might include Torelli-type theorems, strictness results, and restrictions on the monodromy actions (or generalized monodromy actions) for holomorphic families of smooth projective varieties. One might get a nonabelian version of the theory of realizations of motives [32]. Finally, we think that the techniques we use here might also be of

interest in generalizing the geometric Langlands correspondence to higher nonabelian cohomology.

Before saying more about the ideas that we present here, we would like to take note of a few other directions that one could possibly take. It should be possible to remedy the problem of "higher functoriality" by looking directly at Morgan's or Hain's construction and consistently keeping track of what is going on in a suitable $n + 1$-categorical or ∞-categorical way. The work of Hinich [29] would probably be relevant here.

Arapura's treatment of the Green-Lazarsfeld results on jump loci [1] makes explicit use of mixed Hodge structures on dgas and discusses higher homotopy in the sense that he treats the jump loci for higher cohomology. He also has some results for support loci for higher rank local systems [2]. These results are a step in the direction indicated in the previous paragraph.

It might be possible to construct a "big" differential graded algebra which encloses mixed-Hodge theoretic information about higher homotopy, relative to all the representations of π_1. The first and second authors have been thinking about this for some time now and hope to treat it in a future paper.

There is a relationship between simply connected differential graded algebras and simply connected geometric n-stacks, and we have to some extent exploited this in our research on the questions treated in the present paper; however, our calculations in this direction remain largely heuristic. It would be good to understand rigorously the foundations of these calculations; this would help to bridge the gap with the earlier results of Morgan and Hain.

The big differential graded algebra approach might also be related to the notion of higher Malcev completion with its conjectural mixed Hodge structure mentionned at the end of [64]. On the downside, these objects seem to be too "big" (see the example given at the beginning of [66]) and in a certain sense, one could think of the nonabelian cohomology approach as being a way of extracting finite dimensional information from these big objects.

Another approach which uses the notion of differential graded algebra but in a slightly different manner, is Kapranov's "extended moduli space of representations" [35], part of the idea for which apparently comes from Kontsevitch's folkloric "derived deformation theory". Kapranov's extended moduli space visibly contains higher homotopy data, because the tangent complex at a point includes all higher cohomology groups. We have not been able to identify any further the exact nature of this homotopy-theoretic information. It seems quite likely that there should be some type of "mixed Hodge

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structure" on Kapranov's space, and that this would include a mixed Hodge structure on whatever higher homotopy information is contained therein.

For us, an interesting application of Kapranov's idea would be if one could define a notion of "extended n-stack" in order to be able to have an "extended nonabelian cohomology stack" which would extend $\underline{Hom}(X_B, T)$ in the same way that Kapranov's object extends $\underline{Hom}(X_B, BG)$. The reason why this would be helpful is that the extended object could be expected to be formally smooth (this hope seems to be due to Kontsevitch, Deligne, Drinfeld). This smoothness would serve to avoid a large number of the problems which we have with the annihilator ideals in the present paper. Thus, while representing a further abstraction, such a combination of ideas could end up eventually simplifying things.

Yet another direction is that taken recently by Toen [69]. He proposes to do an ∞-categorical version of the idea of Tannaka duality, to view a homotopy type as the Tannaka dual of its ∞-category of local systems of perfect complexes with monoidal structure (tensor product). In this point of view, Toen expects to be able to define Hodge and weight filtrations and to obtain some type of mixed Hodge structure.

To close this part, we note that one of the major advances in mixed Hodge theory, Saito's theory of mixed Hodge modules [53] [54] [55], treats mostly the abelian cohomology theory. It would be good to have a "nonabelian" version of Saito's theory directed toward a notion of Hodge theory for nonabelian cohomology with local coefficients varying constructibly with respect to a stratification. This seems to be a long way off and that could be considered as a long-range goal of any of the above approaches.

Now we get to the ideas that are contained in the present paper. The domain stack X_M which we shall use is an object enclosing X_B, X_{DR} and X_{Dol}, see p. 75 for more details. In the abelian theory the idea of combining together several different types of cohomology theories goes back a long way, see Huber [32] for a compendium. Several people including B. Toen and M. Rosellen had asked us whether there could be any type of "nonabelian motive" taking into account the de Rham, Dolbeault and Betti cohomologies. While there doesn't seem to be anything deep going on here, the notation X_M is a useful bookkeeping device.

Our first preliminary calculations showed that in order to have any hope of having some kind of "mixed Hodge structure" on a nonabelian cohomology stack $\underline{Hom}(X_M, T)$, the coefficient stack T had itself to have a "mixed Hodge structure". Thus our search for the notion of nonabelian mixed Hodge struc-

ture concerned both the coefficient stack and the answer stack. The idea is to obtain a collection of conditions for the definition, such that if T satisfies these conditions then so does $\underline{Hom}(X_M, T)$. Actually the set of conditions which we propose here is just a first attempt; it could in the future turn out to be wrong either by having too many conditions, or by having not enough.

The first and most obvious element is that, in order to integrate the "weight filtration" into the notion of n-stack, we will use the Rees space approach to the notion of filtration: a filtered object is an object over $\mathbf{A}^1$ together with an action of $\mathbf{G}_m$ covering the standard action of $\mathbf{G}_m$ on $\mathbf{A}^1$. If the objects in question are vector spaces, then to a filtered vector space one obtains a corresponding equivariant bundle over $\mathbf{A}^1$ which is essentially its *Rees module.* The origins of the notion of Rees module are actually closely related to the case of "filtered schemes" (cf Fulton [19] pp 90-91, who cites Gerstenhaber [20] and Rees [50], treating filtered rings; these are all related to the "deformation to the normal cone" which we discuss on p. 39 below).

The idea of using the Rees correspondence for the Hodge filtration, and particularly for the Hodge filtration on nonabelian cohomology, is discussed in [56], [60]. The idea goes back to Deligne's approach to the twistor space [15], Deninger's formal version of the Rees correspondence for the Hodge filtration [16], and the Neisendorfer-Taylor deformation between de Rham and Dolbeault cohomology [48]. Perhaps the first place where the notion of Rees bundle appears in connection with the weight filtration, is in Sabbah [52]. Recently (at Irvine, June 1998) Voevodsky made the intriguing remark that the tensor category of mixed motives had two fiber functors, one for the underlying de Rham cohomology and one for the associated-graded of the weight filtration. These clearly correspond to the two points of the stack $\mathcal{A}$ defined in the following paragraph, so this remark can also be seen as a precedent for using the Rees space approach to look at the weight filtration.

Thus we fix the notion of weight filtration as being an equivariant object over $\mathbf{A}^1$, or equivalently an object over the quotient stack

$$\mathcal{A} := \mathbf{A}^1 / \mathbf{G}_m.$$

It follows relatively logically that the notion of a *pair* of filtrations, the weight filtration and the Hodge filtration, is to be interpreted as being an object over the product

$$\mathcal{A}_{wt} \times \mathcal{A}_{hod}$$

where the superscripts indicate separate copies of $\mathcal{A}$, the first one for the weight filtration and the second one for the Hodge filtration. We denote

a geometric n-stack provided with two filtrations, by (V, W, F); this object really means a geometric n-stack

$$Tot^{W,F}(V) \to \mathcal{A}_{wt} \times \mathcal{A}_{hod}.$$

An important element in Hodge theory is the interplay between the Hodge filtration and its complex conjugate. This reappears in the notion of "twistor space" if one wishes to avoid the Hodge filtration altogether. It would certainly be possible to do what we are doing here in the "twistor" context, to obtain a notion of *nonabelian mixed twistor structure*. However, this would add on yet another layer of abstraction which we don't feel is justified at the current time. Thus, for the present purposes, we introduce the notion of *real structure*, which allows us literally to take the complex conjugate of the Hodge filtration for the purposes of measuring the "purity" of the Hodge filtration.

The *associated-graded* of the weight filtration is an object $Gr^W(V)$ obtained by restricting to $[0] \in \mathcal{A}_{wt}$. The "point" represented by the origin is actually the quotient stack $0/\mathbf{G}_m$ with $\mathbf{G}_m$ acting trivially on the point. This quotient stack is the classifying stack denoted $B\mathbf{G}_m$. Thus the origin of $\mathcal{A}$ is in fact the classifying stack $B\mathbf{G}_m$ and the restriction $Gr^W(V)$ is an object over $B\mathbf{G}_m$, which really means an n-stack with action of $\mathbf{G}_m$. Another way to understand this is to think of the total stack as being a $\mathbf{G}_m$-equivariant stack over $\mathbf{A}^1$; since the origin is a fixed point, the fiber over the origin is a $\mathbf{G}_m$-equivariant stack.

The homotopy groups of the associated-graded $Gr^W(V)$ are vector spaces with $\mathbf{G}_m$ action, so they decompose into pieces which one could denote by $\pi_i Gr_k^W(V)$. These are of course the analogues of the classical graded pieces of the weight filtration. They are provided with Hodge filtrations, and real structures, so it makes sense to look at the complex conjugate of the Hodge filtration and ask, for example, that the Hodge filtration be h-opposed with its complex conjugate on $\pi_i Gr_k^W(V)$. Here h is to be chosen according to Deligne's arithmetic of [14] which combines the degree of the weight filtration k and the homotopical degree i: we ask that $h = k - i$ (cf p. 56).

The above ideas are enough to try for a sketch of definition of the notion of nonabelian mixed Hodge structure. We define an object which we call a *pre-namhs* (see p. 50) which basically consists of a geometric n-stack with a bifiltration (V_{DR}, F, W) (a bifiltration is determined by its total stack $Tot^{W,F}(V)$ over $\mathcal{A} \times \mathcal{A}$ cf p. 19), plus a real stack $V_{B,\mathbf{R}}$ with a filtration W and an analytic

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equivalence between (V_{DR}, W) and the complexified $(V_{B,\mathbf{C}}, W)$. The associated graded $Gr^W(V_{DR})$ is a $\mathbf{G}_m$-equivariant geometric n-stack, so (after linearization) it breaks up into components which we denote $Gr_k^W(V_{DR})$. These components have Hodge filtrations F, and analytic real structures coming from their analytic identifications with the complexification of $Gr_k^W(V_{B,\mathbf{R}})$. In particular, the homotopy group sheaves $\pi_i Gr_k^W(V_{DR})$ are actually just complex vector spaces (for $i \geq 2$), and they have Hodge filtrations and real structures. Thus it makes sense to ask for a purity condition relating the Hodge filtration and its complex conjugate (this condition also involves k and i, see below). In fact, we don't put on any such condition in the notion of pre-namhs, because it appears useful to manipulate a notion of bare object not necessarily satisfying any conditions.

This definition of pre-namhs is modelled in an obvious way on Deligne's definitions of mixed Hodge structure and mixed Hodge complex. More generally, if OBJ is any $n+1$-stack (with real structure and analytic extension) then one can define in a similar way a notion of "pre-mixed Hodge object of type OBJ" (cf page 51) as being a collection consisting of a bifiltered object (V_{DR}, W, F) of OBJ, a real filtered object $(V_{B,\mathbf{R}}, W)$, and an analytic equivalence ζ. For $OBJ = VECT$ we recover the notion of pre-mixed Hodge structure, and for $OBJ = PERF$ we recover the notion of pre-mixed Hodge complex. The case of a pre-namhs is obtained from $OBJ = nGEOM$.

The above notion of pre-namhs has an obvious problem, namely that π_1 is only an algebraic group and π_0 is a sheaf of sets. Thus, it isn't clear what type of purity condition to put here, even though the π_0 and π_1 are key parts of what we would like to look at here. Nonetheless, we made some calculations and looked at some examples (in fact, preliminary versions of the computations we do in Part III) and came to the conclusion that there was also a more subtle problem even with respect to the π_i for $i \geq 2$, in the above way of proceeding. In particular, if one tries to define a nonabelian mixed Hodge structure as being a pre-namhs which satisfies some purity conditions on the Hodge filtrations on the homotopy group sheaves, then there seems to be missing the "strictness" which one has come to expect in the mixed Hodge situation.

Our next idea was to look at a specific example where we know that there should be a "nonabelian mixed Hodge structure". The best case we could find was that of S^2. On the one hand, this seems to be an interesting coefficient stack for nonabelian cohomology (cf [65]); the "schematic sphere" had previously been studied in the stable range in [8]. On the other hand, it has

a motivic structure being the homotopy type of $\mathbf{P}^1$ and in fact this "motive" plays an important role for example in Morel's and Voevodsky's theory [44]. In particular, it is clear that whatever the definition of "nonabelian mixed Hodge structure" is, the homotopy type of $\mathbf{P}^1$ should carry one of these.

In doing the calculations, the main thing to recall is the "shift" of the weight filtration by the homotopical degree already mentionned above. This shift first appeared in Deligne's "Hodge III" [14]. Basically there are three numbers involved. The *Hodge degree* which is the integer h such that the Hodge filtration and its complex conjugate are h-opposed (in a more general "mixed twistor" context, h would be the slope of a semistable bundle over the twistor line $\mathbf{P}^1$). The *weight* w is the degree in the weight filtration, equal to k on the graded piece $Gr_k^W(V)$. And the *homotopical degree* i indicates that we are looking at π_i. In a mixed Hodge complex, the homotopical degree is minus the cohomological degree, i.e. one should consider $\pi_i = H^{-i}$. The mixed Hodge condition **MHC** is a linear relationship between these parameters:

$$h = w - i.$$

See our discussion on page 56 for a justification of the sign in this formula.

Now we get back to our calculation for $\mathbf{P}^1$. We have (in complexified homotopy theory)

$$\pi_2(\mathbf{P}^1) = H_2(\mathbf{P}^1) = \mathbf{C},$$

which is of Hodge type $(-1,-1)$. This gives $h(\pi_2) = -2$. The homotopical degree is of course $i(\pi_2) = 2$, and the formula **MHC** determines $w(\pi_2) = 0$. Note of course that the degree w in the weight filtration is the degree in the actual nonabelian filtration we will put on, which is different from the level in the end-result mixed Hodge structure on π_2 exactly because of the formula **MHC**.

On the other hand,

$$\pi_3(\mathbf{P}^1) = \pi_2(\mathbf{P}^1) \otimes_{\mathbf{C}} \pi_2(\mathbf{P}^1),$$

the isomorphism in question being given by the Hopf map or Whitehead product. This has Hodge type $(-2,-2)$ so $h(\pi_3) = -4$. Again of course $i(\pi_3) = 3$ and our formula gives $w(\pi_3) = -1$. Now, in terms of the actual unshifted weight filtration, the Whitehead product map

$$\pi_2 \otimes \pi_2 \to \pi_3$$

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goes from a 1-dimensional space of weight $w = 0$ to a space of weight $w = -1$. In particular, and here is the KEY OBSERVATION:

—the Whitehead products vanish on the associated-graded of the actual (unshifted) weight filtration.

This observation was confirmed by heuristic calculations with differential graded algebras.

It led to the MAIN IDEA: that the associated-graded for the weight filtration in a nonabelian mixed Hodge structure should be an abelian object.

This idea fits in nicely with the whole "quantization" industry (cf `q-alg` or `math.QA`), where the idea of "deformation quantization" is to deform an abelian or classical situation, to a nonabelian one. The weight filtration considered as a $\mathbf{G}_m$-equivariant family of objects parametrized by $\mathbf{A}^1$, may be considered as a deformation of the associated-graded $Gr^W(V)$ which is the fiber over $0 \in \mathbf{A}^1$. It should be possible to explore more deeply the relationship between our notion of weight filtration and the notion of quantization, but we haven't done that.

This parallel raises the interesting question if any of the recent considerable progress in understanding deformation quantization, as exemplified by Kontsevich [39] for example, might be useful in understanding the weight filtration in higher homotopy.

The notion of linearization plays an important role in Toen's higher Tannaka duality [69].

On a technical level, another motivation for our main idea can be found in the construction of the Whitehead product on page 64. The shift by the homotopical degree described above conflicts with the fact that the Whitehead product goes from $\pi_i \otimes \pi_j$ to π_{i+j-1}. Without a supplementary condition, this product would be going between shifted mixed Hodge structures with shifts differing by 1. The condition that the Whitehead products vanish on the associated-graded of the weight filtration means that they descend the weight filtration by one step; this extra step is just what is needed to counteract the extra 1 in the homotopical degree occuring in the Whitehead product (cf p. 66). With this condition, when the mixed Hodge structures on the π_i are shifted back to being actual mixed Hodge structures, the Whitehead product becomes a morphism of mixed Hodge structures. This situation provides a strong motivation for asking that the associated-graded object $Gr^W(\mathcal{V})$ be an abelian object.

To put our main idea into practice, we augment the definition of "pre-namhs" described above, by adding a notion of "linearization". If $\mathcal{V}$ is a

pre-namhs, then one obtains a pre-namhs denoted $Gr^W(\mathcal{V})$, which is split in the sense that the weight filtration comes directly from an action of $\mathbf{G}_m$. A *linearization* of $\mathcal{V}$ is the data of a split pre-mixed Hodge complex $\mathcal{C}$ with an equivalence between $Gr^W(\mathcal{V})$ and the Dold-Puppe pre-namhs of $\mathcal{C}$. (A technical point is that we require $\mathcal{C}$ to be split by a $\mathbf{G}_m$-action splitting the weight filtration, and we require the equivalence to respect splittings.) Without going into all of the definitions (for which we refer to pages 22, 23, 25, 38, 52, 55) we just say here that giving $Gr^W(\mathcal{V})$ the structure of being the Dold-Puppe of a complex is basically the same thing as giving it an "infinite loop-space structure" which means a "linear structure" in an appropriate sense. One effect of this structure is that $\pi_i Gr^W_k(\mathcal{V})$ becomes a complex vector space (with Hodge filtration, real structure) for all $i \geq 0$. The other effect is that the Whitehead products for $Gr^W(\mathcal{V})$ vanish, putting into effect our main idea. We denote the object $\mathcal{C}$ by $LGr^W(\mathcal{V})$, and if such a structure is provided then we say that $\mathcal{V}$ is a *linearized pre-namhs*.

Now the purity condition on a linearized pre-namhs is easy to state: it is just the condition that the pre-mixed Hodge complex $LGr^W(\mathcal{V})$ should be an actual mixed Hodge complex.

This condition is still not enough. It is insufficient on the level of π_0. To understand this part, it will be useful to suppose that the objects we are talking about are schemes rather than n-stacks. The weight filtration is a $\mathbf{G}_m$-equivariant deformation from V to its associated-graded $Gr^W(V)$; on the other hand, the linearization amounts to an equivalence between $Gr^W(V)$ and a complex vector space. In particular, if the deformation is flat then V is smooth (at least locally near where the deformation is taking place). At non-smooth points, we could deform V to, say, its normal cone, but that would then have to be embedded in a vector space. Thus, in general we expect that the weight filtration is not flat over $\mathcal{A}$.

In the case where the weight filtration *is* flat, which means that we are looking locally near some smooth points of V, it seems likely that the "mixed Hodge complex" condition described above, is in fact sufficient to define a good notion of nonabelian mixed Hodge structure. The purity condition on the level of the associated-graded, extends outward in the deformation in the same way as it does for mixed Hodge complexes.

In the non-flat case, we look more carefully at the algebraic geometry of the situation and end up defining the notion of *annihilator ideal*. Since the present introduction is already too long, we won't go into any further details here but refer the reader directly to the text. We just mention that

we isolate three conditions to put on the annihilator ideals denoted **A1**, **A2**, and **A3**. The first two concern only the weight filtration and the third is a compatibility with the split mixed Hodge structure on the associated-graded part. These conditions amount to limitations on the way the non-flat weight filtration can constitute a "jump" when going from $Gr^W(V)$ to V. Condition **A3** complements the purity condition on the associated-graded Gr^W and requires that this condition persist after the jump from $Gr^W(V)$ down to V.

A note of caution about the conditions on the annihilator ideals is in order. We have come up with these conditions only fairly recently in our research on this problem. We feel that the three conditions represent the right collection of conditions, but this is more of an opinion than an established fact. We support this opinion by showing how the whole theory works in a specific computation in Part III, and also give some evidence showing how these conditions already arise in the context of mixed Hodge complexes (cf p. 58). However, it remains possible that in the future it might turn out to be better to modify these conditions in some way (in particular, in order to be able to obtain the various conjectures that we make).

After presenting the definition of nonabelian mixed Hodge structure, we turn to the basic construction where this notion appears, namely the construction of a nonabelian mixed Hodge structure $\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$ when the coefficient object $\mathcal{V}$ is itself a nonabelian mixed Hodge structure. We construct $\mathcal{H}$ as a linearized pre-namhs, when $\mathcal{V}$ is a linearized pre-namhs (modulo a technical hypothesis that the fundamental group object of $\mathcal{V}$ should be represented by a flat linear group scheme). Then we state as Conjecture 1, that if $\mathcal{V}$ is a nonabelian mixed Hodge structure then $\mathcal{H}$ should be one too. We don't prove this conjecture in general; however in Part III we construct a specific pre-namhs $\mathcal{V}$ whose underlying homotopy type is the complexified 2-sphere, and we show that for this specific $\mathcal{V}$, the nonabelian cohomology stack $\mathcal{H}$ is a nonabelian mixed Hodge structure. This example is fairly instructive in showing what is going on.

In the section presenting our conjectures, we also state several other desireable things that should be possible to prove given enough time. We refer the reader there for the statements.

Given: (1) the length necessary just to explain the definition of nonabelian mixed Hodge structure; (2) that any proof of these conjectures would probably take a lot more time to do, a lot more space to write, and be even more highly technical; and (3) the illustrative nature of the example which we can

already give in Part III, we felt that it would be reasonable to leave these main statements as conjectures, and to post what we have done up until now.

So, to conclude this introduction, the current paper is basically just a presentation of the idea of the definition of nonabelian mixed Hodge structure, with a computation of a specific example plus some indications of how the future developpement of this idea might go. Unfortunately, even with this relatively limited goal, the paper is too long.

Part I: Nonabelian weight filtrations

Conventions

We work over the site $\mathcal{Z}$ of schemes of finite type over $\mathbf{C}$ with the etale topology. We look at n-stacks for $0 \leq n < \infty$.

Throughout the paper, all geometric n-stacks (which are n-stacks of n-groupoids over the site $\mathcal{Z}$)—see [62]— will be assumed to be *very presentable* (see [59] [61]); the notation "geometric n-stack" is taken to mean "very presentable geometric n-stack". We also recall that the definition of "geometric" in [62] includes the condition of being "of finite type" in an appropriate sense.

Denote by $nGEOM$ the $n+1$-stack of very presentable geometric n-stacks. This $n+1$-stack includes all not necessarily invertible morphisms, so it is a stack of $n+1$-categories but not of $n+1$-groupoids. One can think of it as being a stack of simplicial categories or "Segal categories" cf [31]. For a scheme Z, the $n+1$-category $nGEOM(Z)$ is by definition the $n+1$-category of very presentable geometric n-stacks $T \to Z$.

This description also works when the base is a stack. In fact, throughout the paper we shall use the equivalence between two different ways of thinking about families of n-stacks: if S is an n-stack of groupoids then the $n+1$-stack of n-stacks over S, i.e.

$$\underline{Hom}(I, nSTACK) \times_{nSTACK} \{S\}$$

(the stack of arrows in $nSTACK$ whose target is S), is equivalent to $\underline{Hom}(S, nSTACK)$. This is discussed in [66]; admittedly that discussion is incomplete but we don't currently wish to add anything to it. Furthermore, we state without proof that if $f : S \to nSTACK$ corresponds to an n-stack $T \to S$, and if S is geometric, then T is geometric if and only if the classifying morphism f lands in $nGEOM$. Thus if S is a geometric n-stack then a morphism $S \to nGEOM$ is the same thing as a geometric n-stack T over S, i.e. a morphism of geometric n-stacks $T \to S$. We shall use this equivalence tacitly throughout the paper.

In the case $n = 0$, a geometric 0-stack is just an Artin algebraic space. If $n = 1$ then a geometric 1-stack is an Artin algebraic 1-stack. The "very presentable" condition corresponds to asking that the stabilizer groups for points of the 1-stack be affine algebraic groups.

Nonabelian filtrations

Fix an $n + 1$-stack OBJ which we think of as parametrizing a "type of object". The typical example is when $OBJ = VECT$ is the 1-stack of vector bundles. This example leads back to the classical notion of filtered vector space. The example which we shall use is when $OBJ = nGEOM$, the $n+1$-stack of geometric n-stacks. Another useful example is $OBJ = nPERF$, the $n + 1$-stack of perfect complexes of length n (see below).

Recall that a *filtered object of OBJ* is a $\mathbf{G}_m$-equivariant morphism from $\mathbf{A}^1$ to OBJ, in other words a morphism of $n + 1$-stacks from (the 1-stack quotient thought of as an $n + 1$-stack) $\mathbf{A}^1/\mathbf{G}_m$, to OBJ.

For the rest of the paper we establish the notation

$$\mathcal{A} := \mathbf{A}^1/\mathbf{G}_m$$

(the quotient 1-stack), and we denote the substacks:

$$\mathbf{G}_m/\mathbf{G}_m \cong * \ \text{ by } \ [1] \subset \mathcal{A},$$

and

$$\{0\}/\mathbf{G}_m \cong B\mathbf{G}_m \ \text{ by } \ [0] \subset \mathcal{A}.$$

We obtain the $n + 1$-stack of filtered objects in OBJ, denoted

$$F.OBJ := \underline{Hom}(\mathcal{A}, OBJ).$$

Note here and throughout, that whenever we write a formula of this kind using Hom or internal $\underline{Hom}$ we make the tacit assumption that the range object has been replaced with a fibrant replacement, cf [63] [31].

The *underlying object* of a filtered object, is the preimage of $[1] \in \mathcal{A}$, call it V. It is a point of OBJ, i.e. a morphism $V : * \to OBJ$ or equivalently a global section $V \in OBJ(Spec(\mathbf{C}))$. If V is a global section of OBJ then a "filtration of V" is the specification of a filtered object in OBJ together with an equivalence between the underlying object and V; we denote such a specification by a letter such as W, and the morphism $\mathcal{A} \to OBJ$ corresponding to W is then denoted $Tot^W(V)$. Introduce also the notation

$$T^W(V) := Tot^W(V)|_{\mathbf{A}^1};$$

in terms of n-stacks over a base this could also be written

$$T^W(V) := Tot^W(V) \times_{\mathcal{A}} \mathbf{A}^1.$$

We denote by $Gr^W(V)$ the *associated-graded object*, which is defined to be the restriction to $[0] \subset \mathcal{A}$. This is an object with an action of $\mathbf{G}_m$, because it is really a morphism $B\mathbf{G}_m \to OBJ$. It is occasionally more convenient to think of $Gr^W(V)$ as being the object itself (i.e. the section $* \to OBJ$ obtained by restriction to the basepoint $* \to B\mathbf{G}_m$), provided with an action of $\mathbf{G}_m$. With all of these notations we denote a filtered object in the usual way by a pair of letters (V, W).

Linearization

Suppose now that $\Phi : OBJ' \to OBJ$ is a functor of $n + 1$-stacks. A Φ-*linearized* (or just *linearized*) filtered object is a filtered object (V, W) together with an object $LGr^W(V) : B\mathbf{G}_m \to OBJ'$ and an equivalence

$$Gr^W(V) \cong \Phi(LGr^W(V)).$$

We denote by $F^\Phi.OBJ$ the $n + 1$-stack of Φ-linearized filtered objects. The terminology "linearization" is due to the fact that we will use this with OBJ' being a stack of "linear objects" (perfect complexes).

Filtered vector spaces

Before getting to the nonabelian examples which are basic to this paper, we start by examining the case of filtered vector spaces. The references for this are [50], [56], [15], [60], [52]. As well as motivating the nonabelian definitions, this discussion is necessary for fixing our conventions regarding the indexing of filtrations.

Apply the above general discussion to the 1-stack $VECT$ of vector bundles: for a scheme X we put $VECT(X)$ equal to the category of vector bundles (of finite rank) on X. A filtered object of $VECT$ is thus a morphism $\mathcal{A} \to VECT$, i.e. a $\mathbf{G}_m$-equivariant vector bundle over $\mathbf{A}^1$. The fiber over $[1]$ is a vector space V, and the total vector bundle is denoted $T^W(V) = Tot^W(V)|_{\mathbf{A}^1}$. An element $v \in V$ gives rise to a unique $\mathbf{G}_m$-invariant section $\tilde{v} = \mathbf{G}_m v$ over $\mathbf{G}_m \subset \mathbf{A}^1$. We say that $v \in W_k V$ if the section $t^k \tilde{v}$ extends to a section of $T^W(V)$ over $\mathbf{A}^1$; i.e. if $\mathbf{G}_m v$ vanishes to order at least $-k$ as a section of $T^W(V)$. Here t is the coordinate vanishing at the origin of $\mathbf{A}^1$. We obtain sub-vector spaces $W_k V \subset V$ which form an *increasing* filtration of V:

$$\ldots W_k V \subset W_{k+1} V \subset \ldots.$$

The fact that $T^W(V)$ is a vector bundle over $\mathbf{A}^1$ means that this filtration is exhaustive (and since V has finite rank, it has only a finite number of nonzero steps). Furthermore, the fiber of $T^W(V)$ over $0 \in \mathbf{A}^1$ is naturally identified with the associated-graded of the actual filtration defined above:

$$T^W(V)|_0 \cong \bigoplus Gr_k^W(V) = \bigoplus W_k V / W_{k-1} V.$$

The identification between $W_k V / W_{k-1} V$ and a subspace of $T^W(V)|_0$ is provided by the map

$$v \mapsto t^k \tilde{v}(0).$$

Now note that $T^W(V)$ is a $\mathbf{G}_m$-equivariant bundle, in particular the fiber over the origin has an action of $\mathbf{G}_m$. To fix conventions for this action (which has an impact on the sign of the indexing integers below), we say that the action is considered as an action on the total space covering the action $\lambda \cdot t = \lambda t$ on $\mathbf{A}^1$. If u is a section of a $\mathbf{G}_m$-equivariant bundle then tu is another section and we have $\lambda \cdot (tu) = \lambda^{-1} t(\lambda \cdot u)$ (the sign can be seen by looking at the trivial action on the trivial bundle with the unit section u). If $\lambda \in \mathbf{G}_m$ then applying the above to the invariant section $\tilde{v}$ we get

$$\lambda \cdot (t^k \tilde{v}) = \lambda^{-k} t^k \tilde{v}.$$

In particular, the image of $W_k V / W_{k-1} V$ is the subspace of $T^W(V)|_0$ on which $\lambda \in \mathbf{G}_m$ acts by the character λ^{-k}. Note that this will motivate our choice of sign in the notations of Theorem 7 and Corollary 8 below.

The above construction gives an equivalence between $F.VECT(Spec(\mathbf{C}))$ and the category of filtered vector spaces. The functor going in the other direction is the "Rees module" construction, see [50], [56], [15], [60], [52].

Filtered vector sheaves

The reader is urged to skip this subsection. Let $VSCH$ denote the 1-stack of vector schemes, and VSH the 1-stack of vector sheaves. Recall that a vector scheme over a base Y is a group scheme which locally looks like the kernel of a map of vector bundles, and a vector sheaf (see [4] [30] [61]) is a sheaf of groups on the big site (Zariski or etale) which locally looks like the kernel of a map of coherent sheaves. We have inclusions

$$VECT \subset VESCH \subset VESH.$$

We consequently obtain notions of "filtered vector schemes" and "filtered vector sheaves". These notions are different but the distinction is somewhat subtle, because the underlying objects are the same: a vector sheaf or vector scheme over $Spec(\mathbf{C})$ is always just a finite dimensional vector space. However, a vector scheme on $\mathbf{A}^1$ is not necessarily a vector bundle, and in turn a vector sheaf is not necessarily a vector scheme. Thus for a vector space V we have three distinct notions of filtration on V, namely a vector bundle filtration; a vector scheme filtration; and a vector sheaf filtration.

In all three cases, the definition of the actual filtration of V by subspaces, using the extension property of sections (i.e. that $v \in W_k V \Leftrightarrow t^k \tilde{v}$ extends to a section of $T^W(V)$), works equally well. Thus, if (V, W) is a filtered vector sheaf then it induces a filtration of V by sub-vector spaces $W_k V$. This construction is not an equivalence of categories, however, and so should be used with care.

We will further discuss these notions when they are necessary; usually they are not. Unless otherwise indicated, a "filtered vector space" will always mean an object of $F.VECT$, i.e. a vector space provided with a filtration in the stack of vector bundles.

Filtered geometric n-stacks

The basic example which we use all the time is when $OBJ := nGEOM$ is the $n + 1$-stack of geometric n-stacks of groupoids (for finite n). In this case, a filtered object (V, W) consists of the specification of a total geometric n-stack $Tot^W(V) \to \mathcal{A}$ together with an equivalence between the fiber over $[1]$ and V; the associated-graded n-stack $Gr^W(V)$ is the fiber over 0 with its $\mathbf{G}_m$-action.

We will study this situation more closely further on below.

Split filtrations

Suppose $U \to B\mathbf{G}_m$ is a $\mathbf{G}_m$-equivariant geometric n-stack (let V be the fiber $V = U \times_{B\mathbf{G}_m} *$; then to be precise V has an action of $\mathbf{G}_m$ and U is the quotient). Then we can define the *filtration of V associated to this action* in the following way. Note that we have a morphism $\mathcal{A} \to B\mathbf{G}_m$ whose fiber is $\mathbf{A}^1$. Set

$$Tot^W(V) := U \times_{B\mathbf{G}_m} \mathcal{A}.$$

The fiber of this over $[1]$ is just the fiber of U over $*$, namely it is V; and the associated-graded or fiber over the origin, is $Gr^W(V) = U \to B\mathbf{G}_m$.

A filtered object obtained from this construction is called a *split* filtered object. However, when we speak of the $n+1$-category of split filtered objects, we usually mean that the splitting is included in the structure, in other words this is just the $n+1$-category of $\mathbf{G}_m$-equivariant objects, and one is conscious that it has a functor towards the category of filtered objects.

If we apply this discussion to the case of filtered vector spaces, we get the standard construction which takes a vector space with $\mathbf{G}_m$-action to the associated filtered vector space, turning the grading into a filtration.

Bifiltered objects

The next example shows why we wrote up the above definition of filtered object in a totally general way: we can apply it to $OBJ := F.nGEOM$, the $n + 1$-stack of filtered geometric n-stacks. We obtain an $n + 1$-stack $F.F.nGEOM$ of "geometric n-stacks having two filtrations". This phrase is in quotation marks because, as we shall see, the datum of an object in $F.F.nGEOM$ is different from the data of an object V provided with two different filtrations W_1 and W_2; the "compatibility between the filtrations" amounts to extra structure.

In fact, let's back up and analyse what it means to have an object in $F.F.OBJ$. We have

$$
\begin{aligned}
F.F.OBJ \ &:= \ \underline{Hom}(\mathcal{A}, F.OBJ) \\
&= \ \underline{Hom}(\mathcal{A}, \underline{Hom}(\mathcal{A}, OBJ)) \\
&= \ \underline{Hom}(\mathcal{A} \times \mathcal{A}, OBJ).
\end{aligned}
$$

Put another way, an object of $F.F.OBJ$ is a $\mathbf{G}_m \times \mathbf{G}_m$-equivariant morphism $\mathbf{A}^2 \to OBJ$. In passing note that there is an action of the symmetric group S_2 on $\mathcal{A} \times \mathcal{A}$, and hence on $F.F.OBJ$; the transposition acts as "interchanging the two filtrations". More generally there is an action of S_m on $F.\ldots F.OBJ$.

Now getting back to the case where $OBJ = nGEOM$, an object of $F.F.nGEOM$ is the same thing as a geometric n-stack $T \to \mathcal{A} \times \mathcal{A}$. As before we shall adopt the notation (V, W, F) for an object of $F.F.nGEOM$, where V is the underlying object and W and F denote the two filtrations. The total stack which "is" the pair of filtrations, shall now be denoted by

$$
Tot^{W,F}(V) \to \mathcal{A} \times \mathcal{A}.
$$

Its fiber over $([1], [1])$ is identified with V. Note that with our notations we have
$$Tot^{W,F}(V) = Tot^F(Tot^W(V)).$$
We denote the fiber of $Tot^{W,F}(V)$ over 0 in the first factor, by $(Gr^W(V), F)$ and the fiber over 0 in the second factor, by $(Gr^F(V), W)$; these are both filtered objects with $\mathbf{G}_m$-action, i.e. morphisms $B\mathbf{G}_m \to F.nGEOM$. The fiber over $(0, 0)$ is denoted $Gr^{W,F}(V)$, and we have

$$Gr^{W,F}(V) \cong Gr^F(Gr^W(V)) \cong Gr^W(Gr^F(V)).$$

It is an object with $\mathbf{G}_m \times \mathbf{G}_m$-action, i.e. a morphism $B\mathbf{G}_m \times B\mathbf{G}_m \to nGEOM$. We shall sometimes call an object (V, W, F) a *bifiltered geometric n-stack*.

We can see the difference between the above notion of bifiltered object, and the notion of the same object having two different filtrations. If (V, G) and (V, H) are two filtered objects having the same underlying object, then we have
$$Tot^G(V) \to \mathcal{A},$$
and
$$Tot^H(V) \to \mathcal{A}.$$
These agree over $[1] \in \mathcal{A}$ so we can glue them together to get a geometric n-stack
$$T \to \mathcal{A} \cup^{[1]} \mathcal{A}.$$
Note that
$$\mathcal{A} \cup^{[1]} \mathcal{A} = (\mathbf{A}^2 - \{0\})/\mathbf{G}_m \times \mathbf{G}_m.$$
Thus T is a $\mathbf{G}_m \times \mathbf{G}_m$-equivariant object over $(\mathbf{A}^2 - \{0\})$. On the other hand we have seen that a bifiltered object means a $\mathbf{G}_m \times \mathbf{G}_m$-equivariant object over $\mathbf{A}^2$. Thus, the extra information needed to go from an object with two separate filtrations, to a bifiltered object, is an extension of the n-stack $T \to (\mathbf{A}^2 - \{0\})$ to an n-stack T' over $\mathbf{A}^2$.

In the case of bifiltered vector spaces, which is the case $OBJ = VECT$, the extension in question is automatic; this corresponds to the "lemme de Zassenhaus" of (Deligne [13], 1.2.1), which says that the double associated-graded taken in one order is equal to the double associated-graded taken in the other order. In our language, an object in $OBJ = VECT$ with two different filtrations is a $\mathbf{G}_m \times \mathbf{G}_m$-equivariant vector bundle over $(\mathbf{A}^2 - \{0\})$,

whereas an object with two filtrations which are compatible in the sense that they satisfy the "lemme de Zassenhaus" is a $\mathbf{G}_m \times \mathbf{G}_m$-equivariant vector bundle over $\mathbf{A}^2$. Recall that every algebraic vector bundle over $(\mathbf{A}^2 - \{0\})$ extends uniquely to a vector bundle over $\mathbf{A}^2$ (this is proved by extending the algebraic vector bundle to a reflexive sheaf by taking the direct image from the open set to all of $\mathbf{A}^2$, and then using the theorem that reflexive sheaves on smooth surfaces are bundles). The common double associated-graded is just the fiber over $(0,0)$ of this extension. The fact that one doesn't have a "lemme de Zassenhaus" for three filtrations translates the fact that the extension property for vector bundles doesn't hold for $(\mathbf{A}^3 - \{0\})$.

It would be interesting to be able to say something about the problem of existence and/or uniqueness extensions of families of other types of objects (schemes, geometric n-stacks, perfect complexes) from $(\mathbf{A}^2 - \{0\})$ to $\mathbf{A}^2$. While the "lemme de Zassenhaus" probably fails in all of these contexts, it would be good to understand just to what extent it fails.

Bifiltrations and linearization

We finish with a word about bifiltrations where one of the filtrations is linearized. In practice below we will work with "weight filtrations" which are filtered objects (V, W) provided with linearization via a certain functor Φ as envisaged above. We will then want to look at filtered objects in $F^\Phi.nGEOM$ (the second filtration being the "Hodge filtration").

We will still denote objects of $F.F^\Phi.nGEOM$ as triples (V, W, F), where (V, W) is an linearized filtered object and F is a filtration of (V, W) considered as an object of $F^\Phi.nGEOM$. Concretely (and denoting for the time being the linearization functor by $\Phi : OBJ' \to nGEOM$) this means that the graded object $(Gr^W(V), F)$ is linearized along Φ, i.e. it is the image of a filtered object $(LGr^W(V), F)$ of OBJ' (and also it has the $\mathbf{G}_m$-action because of being the associated-graded for W) or in other words,

$$(LGr^W(V), F) : B\mathbf{G}_m \to F.OBJ'(Spec(\mathbf{C})),$$

with given an equivalence

$$\Phi(LGr^W(V), F) \cong (Gr^W(V), F)$$

of $\mathbf{G}_m$-equivariant objects in $F.nGEOM$.

(Hemi-)perfect complexes and Dold-Puppe linearization

In this section we look at the specific linearization functor Φ which will interest us. It is the forgetful functor from geometric n-stacks of spectra, to geometric n-stacks. Equivalently, it is the Dold-Puppe functor from perfect n-complexes, to geometric n-stacks. Let $nPERF$ denote the $n+1$-stack of perfect complexes of length n (supported in the interval $[-n, 0]$). Recall from [31] that this may be defined in the following way. For any scheme $Z \in \mathcal{Z}$ we put $nPerf(Z)$ equal to the category of complexes of sheaves [4] on Z which are locally quasiisomorphic to complexes of vector bundles of length n (supported in $[-n, 0]$, say). Then set $nPERF(Z)$ equal to the simplicial (or Segal) category obtained by localizing $nPerf(Z)$ by dividing out by quasiisomorphisms, in the sense of Dwyer-Kan [17]. This simplicial category is in fact $n+1$-truncated, i.e. it may be considered as an $n+1$-category. It is proved in [31] that the resulting $n+1$-prestack is an $n+1$-stack.

The functor

$$DP : nPERF \to nGEOM$$

is just the Dold-Puppe functor [33] [62].

Another way of defining the $n+1$-stack $nPERF$ is as follows. Fix $N > n$. Then the $n + N + 1$-substack

$$(n+N)GEOM^{ptd,N-conn} \subset (n+N)GEOM$$

consisting of pointed, N-connected geometric (and by our convention, very presentable) $n + N$-stacks, is in fact $n + 1$-truncated. (Note that an object of this substack, over a scheme Z, is an $n + N$-stack $T \to Z$ provided with a section $t : Z \to T$ and such that T is relatively N-connected over Z.) This substack is equivalent to $nPERF$. The equivalence in one direction is obtained by taking a perfect complex supported in $[-n, 0]$ and shifting it to $[-n - N, -N]$ then applying Dold-Puppe. In the other direction, take the singular cochain complex for example. The fact that these two constructions are inverses is the fact that rational homotopy theory in the stable range is equivalent to the theory of complexes of rational vector spaces [43, Ch.8§1, Theorem 7]. (We leave it to the reader to fill in the details of the proof of this equivalence).

---

[4] By [61] it doesn't matter whether we work with complexes of sheaves of $\mathcal{O}$-modules, or complexes of sheaves of abelian groups, since we are working on the big site.

In this second point of view, the functor

$$DP : nPERF = (n + N)GEOM^{ptd, N-conn} \to nGEOM$$

is the functor which to an object (T, t) associates the N-th iterated loop stack of T at the basepoint t.

Example: A perfect complex over $Spec(\mathbf{C})$ is just a complex of $\mathbf{C}$-vector spaces. In this case, it is non-canonically quasiisomorphic to a complex with zero differential (the direct sum of its cohomology spaces) so its Dold-Puppe decomposes as a product of things of the form $K(\mathbf{C}^{a_i}, i)$.

Hemiperfect complexes

Somewhat unfortunately, we will sometimes have to deal with a notion slightly more general than that of "perfect complex" which we call "hemiperfect complex". This notion appears, for complexes of length 1, in [41] for example. The origin of the problem is that if $K^\cdot$ is a perfect complex (over a scheme Z, say) supported in the interval $[-n, \infty)$ then its truncation $C^\cdot = \tau_{\leq 0}(K^\cdot)$ is no longer a perfect complex. Rather, the $C^i = K^i$ are vector bundles for $i < 0$ but

$$C^0 = \ker(K^0 \xrightarrow{d} K^1)$$

is only a vector scheme.

Thus we introduce the following definition: a *hemiperfect complex of length n* over a scheme Z is a complex of sheaves of groups on the big etale site over Z, which is locally quasiisomorphic to a complex of the form $C^\cdot$ where C^i is zero outside the interval $[-n, 0]$, where the C^{-i} are vector bundles for $-n \leq -i \leq -1$, and where C^0 is a vector scheme. Let $nHPerf(Z)$ denote the 1-category of such objects, and let $nHPERF(Z)$ denote the Dwyer-Kan localization dividing out by quasiisomorphisms. We state without proof that:
—the Segal category $nHPERF(Z)$ is $n + 1$-truncated, so it may be considered as an $n + 1$-category;
—if K and L are hemiperfect complexes, then the morphism n-groupoid between K and L in $nHPERF(Z)$ is calculated by the formula

$$nHPERF(Z)_{1/}(K, L) = DP(\tau_{\leq 0}(\mathbf{R}Hom(K, L)))$$

where the $\mathbf{R}Hom$ is taken in the derived category of complexes of sheaves of abelian groups or sheaves of $\mathcal{O}$-modules on the big Zariski site (we state

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that the two answers are the same); and

—the $n + 1$-prestack $Z \mapsto nHPERF(Z)$ is actually an $n + 1$-stack on the site $\mathcal{Z}$.

These statements are the analogues of statements proved for perfect complexes in [31], see also [61] for the fact that the $\mathbf{R}Hom$ may be taken in complexes of sheaves of abelian groups or sheaves of $\mathcal{O}$-modules. We assume that the same proofs work for hemiperfect complexes.

If B is an n-stack then a *hemiperfect complex over Z* is defined as being a morphism $B \to nHPERF$.

It is sometimes useful to have a second point of view on perfect complexes, for hemiperfect complexes. This is somewhat problematic, because a simple transcription of that discussion cannot work. Indeed, an $n + N$-stack which is geometric and also N-connected relative to the base scheme Z is automatically smooth and corresponds to a perfect complex. In particular, it cannot come from a hemiperfect complex which is not perfect. Thus we have to use the notion of very presentable n-stack.

A hemiperfect complex $\mathcal{C}$ may be viewed as a relatively pointed very presentable $n + N$-stack $(\tilde{\mathcal{C}}, p)$ over Z (i.e. p is a section of $\tilde{\mathcal{C}}/Z$) such that $\tilde{\mathcal{C}}$ is relatively N-connected, and such that the N-th looping $\Omega^N(\tilde{\mathcal{C}}, p)$ is geometric. If furthermore $\Omega^N(\tilde{\mathcal{C}}, p)$ is *smooth* relative to Z, then $\tilde{\mathcal{C}}$ itself will be geometric and this will correspond to a perfect complex.

Our statement of the equivalence between the notion of truncated perfect complex and the notion of $(\tilde{\mathcal{C}}, p)$ as above, is given with only a sketch of proof: starting with $\mathcal{C}$ we set $\tilde{\mathcal{C}}$ to be the Dold-Puppe of the shift of $\mathcal{C}$ by N places to the left. On the other hand given $\tilde{\mathcal{C}}$ we construct $\mathcal{C}$ as the relative homology complex of $\tilde{\mathcal{C}}$ over Z, which is then shifted by N places to the right. The proof that if $\mathcal{C}$ is hemiperfect then $\Omega^N(\tilde{\mathcal{C}}, p)$ is geometric, is the same as the proof for perfect complexes given in [62]. The proof in the other direction, that if $\Omega^N(\tilde{\mathcal{C}}, p)$ is geometric then $\mathcal{C}$ is hemiperfect, is obtained by looking at $\mathcal{C}$ as the relative tangent complex of $\Omega^N(\tilde{\mathcal{C}}, p)$ over Z.

We still obtain a Dold-Puppe functor which we again denote by

$$DP : nHPERF \to nGEOM.$$

It sends $\mathcal{C}$ to the geometric n-stack $\Omega^N(\tilde{\mathcal{C}}, p)$, or equivalently it may be viewed directly as the Dold-Puppe of $\mathcal{C}$ see [33].

Cohomology and homotopy sheaves

If $C^{\cdot}$ is a (hemi)perfect complex of length n on a scheme Z then its $-i$-*th cohomology* is a vector sheaf denoted $H^{-i}(C^{\cdot})$ over Z, nonzero for $0 \le i \le n$ (see [30]). We have the following compatibility with the relative homotopy sheaves of the Dold-Puppe:

$$\pi_i(DP(C^{\cdot})/Z) = H^{-i}(C^{\cdot}).$$

Thus, the $H^{-i}(C^{\cdot})$ plays the role of the homotopy sheaf π_i of the perfect complex.

In terms of the second point of view given above, if we think of $C^{\cdot}$ as corresponding to a relatively N-connected very presentable $n + N$-stack $\tilde{C}$ over Z, then $H^{-i}(C^{\cdot})$ may be defined as being equal to the relative homotopy group sheaf $\pi_{N+i}(\tilde{C}/Z)$. In these terms, the above compatibility becomes tautological.

Dold-Puppe linearization

We now say that a *Dold-Puppe-linearized filtered geometric n-stack* is an object
$$(V, W) \in F^{DP}.nGEOM,$$

i.e. it is a filtered geometric n-stack with extra structure on the associated-graded via the functor

$$DP : nHPERF \to nGEOM.$$

Concretely, this means that (V, W) is a quintuple

$$(V, W) = (V, \epsilon, Tot^W(V) \to \mathcal{A}, \eta, LGr^W(V))$$

where V is a geometric n-stack; $Tot^W(V)$ is a geometric n-stack over $\mathcal{A}$; ϵ is an equivalence
$$\epsilon : V \cong Tot^W(V) \times_{\mathcal{A}} [1];$$

where
$$LGr^W(V) : B\mathbf{G}_m \to nHPERF$$

is a hemiperfect complex with $\mathbf{G}_m$-action; and where η is an equivalence between DP of this latter, and the fiber of $Tot^W(V)$ over 0:

$$\eta : DP(LGr^W(V)) \cong Tot^W(V) \times_{\mathcal{A}} B\mathbf{G}_m.$$

We denote by $LF.nGEOM := F^{DP}.nGEOM$, the $n+1$-stack of Dold-Puppe-linearized filtered geometric n-stacks.

If the object $LGr^W(V)$ is a perfect complex then we say that the linearization is a *perfect linearization*. In the general case we may sometimes call it a *hemiperfect linearization* in order to avoid confusion, but if no specification is made then the linearization is *a priori* only hemiperfect.

The compatibility between cohomology groups of a complex and homotopy groups of the Dold-Puppe, gives in the case of a linearized filtration the formulae

$$\pi_i Gr^W(V) = H^{-i}(LGr^W(V)).$$

In particular, even $\pi_0 Gr^W(V)$ and $\pi_1 Gr^W(V)$ are vector sheaves (i.e. vector spaces when the base is a point).

Further study of filtered n-stacks

If (V, W) is a filtered geometric n-stack then

$$T^W(V) := Tot^W(V) \times_{\mathcal{A}} \mathbf{A}^1$$

is a geometric n-stack mapping to $\mathbf{A}^1$. This situation leads to a certain amount of algebraic geometry which is basically the same as the geometry of a scheme mapping to $\mathbf{A}^1$. In studying this, it is important to recall [62] that there is a scheme of finite type Z with a surjective smooth map

$$Z \to T^W(V).$$

Coherent sheaves on geometric n-stacks

Before getting to our geometric study of $T^W(V)$ or $Tot^W(V)$, we digress for a moment to define the abelian category $Coh(Tot^W(V))$ of coherent sheaves. Let COH be the 1-stack (in additive categories) which to each scheme Y assigns the additive category of coherent sheaves on Y. If X is a geometric n-stack, put $Coh(X) := \underline{Hom}(X, COH)$.

Lemma 1 *If X is a geometric n-stack, then $Coh(X)$ is an abelian category containing a canonical object $\mathcal{O}$. If $f : Z \to X$ is a flat morphism of geometric n-stacks, then the induced morphism $Coh(X) \to Coh(Z)$ is exact. If f is flat and surjective then this morphism is an exact inclusion of abelian categories.*

Proof: The proof is by induction on n. Note that if X is a scheme then $Coh(X) = COH(X)$ is the usual abelian category of coherent sheaves, and smooth morphisms between schemes induce exact pullback functors. The case of schemes takes care of the case $n = -1$ in a certain sense in the induction, so we don't bother starting with $n = 0$. Suppose the statement is known for $n - 1$. Suppose X is a geometric n-stack and choose a smooth surjection from a scheme $Z \to X$. By induction, we know that

$$Coh(Z), \quad Coh(Z \times_X Z) \text{ and } Coh(Z \times_X Z \times_X Z)$$

are abelian categories, and that the pullback functors p_i and p_{ij} between them are exact embeddings of abelian categories. Now an object $\mathcal{F}$ of $Coh(X)$ is the same thing as a pair $(\mathcal{G}, g)$ where $\mathcal{G} \in Coh(Z)$ and $g : p_1^*(\mathcal{G}) \cong p_2^*(\mathcal{G})$ is an isomorphism such that the composition of $p_{12}^*(g)$ and $p_{23}^*(g)$ is equal to $p_{13}^*(g)$. From this description and the fact that the functors p_i and p_{ij} involved are exact, it is clear that $Coh(X)$ admits kernels, cokernels, and that the coimages are equal to the images; i.e. $Coh(X)$ is abelian. Furthermore it is clear that the pullback functor $Coh(X) \to Coh(Z)$ is a faithful and exact.

The pair $(\mathcal{O}, 1)$ corresponds to the object $\mathcal{O}$ of $Coh(X)$.

Suppose now that $f : X \to X'$ is a flat morphism (resp. flat surjection). Let $Z' \to X'$ be a smooth surjection from a scheme. Then $X \times_{X'} Z' \to X$ is a smooth surjection. Let $Z \to X \times_{X'} Z'$ be a smooth surjection from a scheme, so $Z \to X$ is a smooth surjection from a scheme. Note that $Z \to Z'$ is a flat morphism (resp. flat surjection) of schemes. In the diagram

$$
\begin{array}{ccc}
Coh(X') & \to & Coh(X) \\
\downarrow & & \downarrow \\
Coh(Z') & \to & Coh(Z)
\end{array}
$$

the vertical arrows are exact and faithful, and the bottom arrow is exact (resp. exact and faithful). Therefore the top arrow is exact (resp. exact and faithful). |||

This lemma implies, among other things, that the objects of $Coh(X)$ are noetherian.

For any n-stack X one can define an additive category

$$Coh(X) := \underline{Hom}(X, COH).$$

It doesn't look likely that $Coh(X)$ will be abelian for an arbitrary n-stack X, but we don't have a counterexample. Note however that $Coh(X)$ depends

only on the 1-truncation $\tau_{\leq 1}(X)$ so any n-stack whose 1-truncation is equivalent to the 1-truncation of a geometric n-stack, will yield an abelian category $Coh(X)$.

Geometric study of $Tot^W(V)$

We can now continue with our study of $T^W(V)$ and $Tot^W(V)$. To start with, we say that (V, W) is *flat* if $T^W(V) \to \mathbf{A}^1$ is a flat morphism. This means that the morphism of schemes $Z \to \mathbf{A}^1$ is flat.

We obtain from the previous lemma an abelian category $Coh(T^W(V))$ of coherent sheaves on the geometric n-stack $T^W(V)$. There is a canonical object, the structure sheaf $\mathcal{O} \in Coh(T^W(V))$. On the other hand, let t denote the coordinate on $\mathbf{A}^1$. This pulls back to a section of the structure sheaf, which we can view as a morphism in the category $Coh(T^W(V))$, eventually raised to a power:

$$t^m : \mathcal{O} \to \mathcal{O}.$$

Let $\ker(t^m)$ denote the subobject of $\mathcal{O}$ which is the kernel of t^m. It is a coherent sheaf of ideals (i.e. a subobject of the structure sheaf) on $T^W(V)$. We have

$$\ker(t^m) \subset \ker(t^{m+1}).$$

Since $\mathcal{O}$ is noetherian (look over any scheme in $\mathcal{Z}$ with a smooth surjection to $T^W(V)$), this increasing sequence of ideals eventually becomes stationary.

We now note that these ideals are "$\mathbf{G}_m$-equivariant" i.e. they actually come from ideals (for which we use the same notation) $\ker(t^m)$ on $Tot^W(V) = T^W(V)/\mathbf{G}_m$. To see this, let $\mathcal{O}\{1\}$ be the structure sheaf $\mathcal{O}$ but with action of $\mathbf{G}_m$ twisted by the standard character of $\mathbf{G}_m$; it is an object in $Coh(Tot^W(V))$. The coordinate t is a morphism

$$t : \mathcal{O} \to \mathcal{O}\{1\}$$

and its powers are morphisms

$$t^m : \mathcal{O} \to \mathcal{O}\{m\}.$$

Thus $\ker(t^m)$ has a meaning on $Tot^W(V)$.

These ideals are related to the canonical flat sub-filtration of a filtered object, defined in the following lemma:

Lemma 2 *Suppose (V, W) is a filtered geometric n-stack. Then the $n+1$-category of objects $(V, W') \to (V, W)$ (filtrations on V mapping to W) such that (V, W') is flat, has a final object (V, W^{fl}), which we call the* canonical flat subfiltration. *Furthermore, $Tot^{W^{\mathrm{fl}}}(V)$ is the closed geometric substack of $Tot^W(V)$ defined by the coherent sheaf of ideals $\ker(t^m)$ (for $m \gg 0$).*

Proof: Define (V, W^{fl}) by the condition that $Tot^{W^{\mathrm{fl}}}(V)$ be the closed geometric substack of $Tot^W(V)$ defined by the coherent sheaf of ideals $\ker(t^m)$ (for $m \gg 0$). Note that for $m \gg 0$ this sequence of ideals becomes stationary, as can be verified by going to a scheme Z with smooth surjection to $Tot^W(V)$. Note also that if Z^{fl} is the closed subscheme of Z defined by the sheaf of ideals $\ker(t^m)$ on Z (which is also the pullback of $\ker(t^m)$ from $Tot^W(V)$) then $Z^{\mathrm{fl}} \to Tot^{W^{\mathrm{fl}}}(V)$ is a smooth surjection.

The coordinate t is not well-defined on $\mathcal{A}$ but on any scheme mapping to $\mathcal{A}$ we can locally chose a definition of t. On overlaps of neighborhoods of definition, the coordinates t differ by invertible elements. Note that $\ker(t^m)$ on Z, for $m \gg 0$, is the ideal of t-torsion in $\mathcal{O}(Z)$. In particular, $\mathcal{O}(Z^{\mathrm{fl}})$ has no t-torsion so the map $Z^{\mathrm{fl}} \to \mathcal{A}$ is flat.

Suppose that (V, W') is a flat filtration mapping to (V, W), and let Z' be a scheme with a smooth surjection to $Tot^{W'}(V)$, so that $Z' \to \mathbf{A}^1$ is flat. We may assume that there is a lifting to a map $Z' \to Z$. Then $\mathcal{O}(Z')$ has no t-torsion, so the t-torsion in $\mathcal{O}(Z)$ goes to zero in $\mathcal{O}(Z')$. Thus, the map factors through $Z' \to Z^{\mathrm{fl}}$. This implies that the map

$$Tot^{W'}(V) \to Tot^W(V)$$

factors through a map

$$Tot^{W'}(V) \to Tot^{W^{\mathrm{fl}}}(V).$$

The factorization is unique since

$$Tot^{W^{\mathrm{fl}}}(V) \subset Tot^W(V)$$

is a closed substack defined by a sheaf of ideals. This completes the proof. |||

Next we note that $Coh(Gr^W(V))$ is equivalent to the category of objects in $Coh(Tot^W(V))$ on which t acts as zero. The operation of restriction of a

coherent sheaf from $Tot^W(V)$ to $Gr^W(V)$ corresponds to tensoring by $\mathcal{O}_{Gr} := \mathcal{O}/t\mathcal{O}\{-1\}$. If $\mathcal{F}$ is a coherent sheaf on $Tot^W(V)$ then

$$\mathcal{F}|_{Gr^W(V)} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_{Gr}.$$

All of these statements can be seen by using Lemma 1 to reduce to the case of a scheme Z surjecting smoothly to $Tot^W(V)$.

Finally we define the *annihilator ideals* of (V, W) to be the images

$$Ann^W(t^m; V) := im(\ker(t^m) \otimes_{\mathcal{O}} \mathcal{O}_{Gr} \to \mathcal{O}_{Gr}).$$

These are ideals on $Gr^W(V)$, in an increasing sequence,

$$Ann^W(t^m; V) \subset Ann^W(t^{m+1}; V)$$

and this sequence eventually becomes stationary (because $\mathcal{O}_{Gr}$ is noetherian).

Lemma 3 *Suppose in the above situation, that for all $m \geq 1$ we have*

$$Ann^W(t; V) = Ann^W(t^m; V).$$

Then the $\ker(t^m)$ are all equal, and they are already supported on $Gr^W(V)$, i.e. we have
$$\ker(t^m) = Ann^W(t; V).$$

Proof: It suffices to prove this for a morphism of schemes $Z \to \mathbf{A}^1$ (for the ideals $\ker(\ldots)$ and $Ann(\ldots)$ defined in the same way on Z). To see that this suffices, let Z be a scheme with a smooth surjection

$$Z \to Tot^W(V) \times_{\mathcal{A}} \mathbf{A}^1,$$

and note also that Z has a smooth surjection to $Tot^W(V)$. The ideals defined relative to the map $Z \to \mathbf{A}^1$ are the pullbacks of the corresponding ideals on $Tot^W(V)$, and since $Z \to Tot^W(V)$ is a smooth surjection, the formulae in question may be proven after pullback to Z.

Now the question for $Z \to \mathbf{A}^1$ is just a statement in usual algebraic geometry. To justify it, note that we have an exact sequence

$$0 \to \mathcal{T} \to \mathcal{O}(Z) \to \mathcal{F} \to 0$$

where $\mathcal{T}$ is the submodule of t-torsion in $\mathcal{O}(Z)$, and $\mathcal{F}$ is flat over a neighborhood of $0 \in \mathbf{A}^1$. Note that the torsion is of finite length, in other words

$\mathcal{T}$ is really a $\mathbf{C}[t]/t^M$-module for some M. Then $\ker(t^m)$ and $Ann(t^m, Z)$ depend only on $\mathcal{T}$ (these objects may be defined for any $\mathbf{C}[t]$-module, and the statement is that the objects for $\mathcal{T}$ coincide with those for $\mathcal{O}(Z)$). On the other hand, the standard proof of the Jordan normal form works even in this context where the vector spaces are of infinite dimension but the order of nilpotency is bounded, so we may write

$$\mathcal{T} = \bigoplus M_i, \quad M_i = \mathbf{C}[t]/t^{e_i}.$$

On a piece M_i, direct calculation shows that $Ann(t^m; M_i)$ is zero if $m < e_i$ and is $\mathbf{C}$ if $m \geq e_i$. Thus the condition that $Ann(t^m; M_i) = Ann(t; M_i)$ implies that $e_i = 1$. On the pieces M_i with $e_i = 1$ (which are just of the form $M_i = \mathbf{C}$) one immediately gets that $\ker(t^m)$ is independent of m and that these are equal to $Ann(t, M_i)$.

In this proof, the condition of the lemma (which we shall call **A1** below) is seen as equivalent to the statement that the t-torsion in $\mathcal{O}(Z)$ is actually a $\mathbf{C} = \mathbf{C}[t]/(t)$-module. |||

Conditions A1 and A2

We now state the *first and second conditions on the annihilator ideals* that will be used (for the weight filtration) in the definition of nonabelian mixed Hodge structure:

A1(V, W) This condition says that for all $m \geq 1$ we have

$$Ann^W(t; V) = Ann^W(t^m; V).$$

This says that the conclusion of the previous lemma applies.

If this condition holds, then $Ann^W(t; V)$ is an ideal on $Tot^W(V)$, and the quotient by this ideal is the structure sheaf for the canonical flat extension $Tot^{W^{\text{fl}}}(V)$.

A2(V, W) This condition says that $Ann^W(t, V) \subset \mathcal{O}_{Gr^W(V)}$ is contained in the ideal of $0 \in Gr^W(V)$ (the point 0 being defined because of the structure of complex, giving a structure of vector space on $\pi_0 Gr^W(V)$).

Corollary 4 *If (V, W) satisfies conditions* **A1**(V, W) *and* **A2**(V, W) *then there is a quasi-finite flat cover $Y \to \mathbf{A}^1$, containing a lift of the origin, such that the morphism $Tot^W(V) \times_{\mathcal{A}} Y \to Y$ admits a section.*

Proof: Let (V, W^{fl}) denote the canonical flat subfiltration. Our conditions **A1** and **A2** imply that for $m \gg 0$ the ideals $\ker(t^m)$ (which are all equal) restricted to the fiber of the total space over the origin, are not (1). This ideal defines $Tot^{W^{\mathrm{fl}}}(V)$, in particular $Gr^{W^{\mathrm{fl}}}(V)$ is nonempty. To finish the proof of the corollary, we may now assume that (V, W) is a filtered object with $Tot^W(V)$ flat over $\mathcal{A}$ and such that $Gr^W(V)$ is nonempty. Thus

$$T^W(V) \to \mathbf{A}^1$$

is a flat morphism whose fiber over 0 is nonempty. Choose a smooth surjection from a scheme $Z \to T^W(V)$ so that $Z \to \mathbf{A}^1$ is a flat morphism of schemes of finite type. In this situation it is automatic that there is $Y \to \mathbf{A}^1$ as in the statement, such that there exists a section $Y \to Z$. $|||$

This corollary says that, if conditions **A1** and **A2** hold, then $Gr^W(V)$ is not "totally detached" from V.

Counterexample (1): A simple construction gives an example why we need some condition of type **A2** to insure that $Gr^W(V)$ isn't detached from V. Let V be a scheme, and let $U \to B\mathbf{G}_m$ be any scheme with $\mathbf{G}_m$ action. Then set
$$Tot^W(V) := V \coprod U$$
with its map to $\mathcal{A}$ (sending V to $[1]$ and U to the $B\mathbf{G}_m$ which is the origin). This defines a perfectly good filtration (V, W) in which $Gr^W(V) = U$. However, U doesn't really have anything to do with V. In this construction, the ideals $\ker(t^m)$ and $Ann^W(t^m; V)$ are all equal to the unit ideal (1) of $\mathcal{O}_U$. The canonical flat extension (V, W^{fl}) is the filtration whose total space is just V and whose associated-graded is empty.

Counterexample (2): One cannot conclude, in the above corollary, existence of a section over $\mathbf{A}^1$ (and certainly not existence of an equivariant section either) as shown by the following example. Let $f : Y \to \mathbf{A}^1$ be a double cover ramified at the origin, and let $T := Y \times \mathbf{G}_m$. Define a morphism $p : T \to \mathbf{A}^1$ by $p(y, t) := tf(y)$. This morphism is $\mathbf{G}_m$-equivariant (by construction) so it defines a morphism $Y = T/\mathbf{G}_m \to \mathcal{A}$, in other words we get a filtered object (V, W) with V equal to the fiber of T over $1 \in \mathbf{A}^1$ (which is also equal to the inverse image of $\mathbf{G}_m \subset \mathbf{A}^1$ in Y) and with $Tot^W(V) := T/\mathbf{G}_m = Y$. We have

$$Tot^W(V) \times_{\mathcal{A}} \mathbf{A}^1 = T \xrightarrow{p} \mathbf{A}^1.$$

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The fiber of T over the origin has multiplicity two, i.e. the morphism p is ramified along the fiber; this implies that there doesn't exist a section over $\mathbf{A}^1$ (or over any etale neighborhood of the origin).

Filtered and $\mathbf{G}_m$-equivariant perfect complexes

We can apply our discussion of filtered objects, to the stack $nPERF$ of length-n perfect complexes, to obtain an $n+1$-stack $F.nPERF$ of filtered perfect complexes. The same holds for hemiperfect complexes, and we get an $n+1$-stack $F.nHPERF$ of filtered hemiperfect complexes.

We leave for another place the task of establishing the relationship between filtered complexes in the usual sense (cf Illusie [33]) for example, and objects of $F.nPERF$ or $F.nHPERF$.

Delooping and truncation

Suppose $\mathcal{C}$ is a complex of sheaves supported in $[-n, 0]$. Let $\Omega^k \mathcal{C}$ denote the complex obtained by shifting $\mathcal{C}$ to the right by k places, and doing the "intelligent" truncation in degrees ≤ 0 which consists of replacing the term of degree zero by the kernel of its differential to the term of degree one. Thus $\Omega^k \mathcal{C}$ is supported in the interval $[k-n, 0]$. If $k > n$ this produces the zero complex. We call this complex the *k-th loop complex of* $\mathcal{C}$. This terminology is due to the following compatibility with the Dold-Puppe operation:

$$DP(\Omega^k \mathcal{C}) \cong \Omega^k(DP\mathcal{C}, 0)$$

where the terminology on the right means taking k times the loop space based at the zero section of $DP(\mathcal{C})$. The equivalence is a canonical weak homotopy equivalence of $n-k$-stacks.

If $\mathcal{C}$ is a hemiperfect complex then $\Omega^k \mathcal{C}$ is again hemiperfect. The operation Ω combines a shift and a truncation, so if $\mathcal{C}$ is perfect, the loop complex might be hemiperfect without being perfect. In fact this is the main motivation for introducing hemiperfect complexes: they arise as the truncations of perfect complexes.

Lemma 5 *Suppose $(\mathcal{C}, F)$ is a filtered hemiperfect complex. Suppose that the cohomology sheaves $H^{-i}(Tot^F(\mathcal{C})/\mathcal{A})$ are vector bundles over $\mathcal{A}$. Then there exists a splitting, namely an equivalence in $F.nHPERF$ between $(\mathcal{C}, F)$ and $\bigoplus_i (H^{-i}(\mathcal{C}), F)$. In particular $(\mathcal{C}, F)$ is a filtered perfect complex, as*

are all of the loop complexes $(\Omega^k \mathcal{C}, F)$. *For filtered perfect complexes the condition (which we denote by* **Str***) that the* H^{-i} *be vector bundles over* $\mathcal{A}$*, is equivalent to strictness of the differentials or equivalently degeneration of the spectral sequence at* E_1*. Finally, strictness of the differentials implies that the cohomology vector sheaves are vector bundles.*

Proof: Suppose the cohomology vector sheaves are vector bundles. The total complex $Tot^F(\mathcal{C})$ is classified by a collection of extension classes in higher Ext groups between the cohomology vector sheaves. If the vector sheaves are vector bundles then these Ext groups coincide with the usual ones, and in fact are calculated cohomologically: $Ext^i(U, V) = H^i(\mathcal{A}, U^* \otimes V)$. But $\mathcal{A}$ is cohomologically trivial for vector bundle coefficients (use the principal $\mathbf{G}_m$-bundle $\mathbf{A}^1 \to \mathcal{A}$ to see this). Thus the higher Ext groups vanish which implies that the complex splits.

If the complex splits, it is clear that the spectral sequence degenerates. This latter condition is equivalent by Deligne [13] Proposition 1.3.2, to strictness of the differentials; and finally, strictness of the differentials means that the differentials of the total complex are strict morphisms of vector bundles, which in turn implies that the cohomology vector sheaves are vector bundles. |||

If $(\mathcal{C}, W)$ is a filtered hemiperfect complex then we can replace the vector scheme $(Tot^W(\mathcal{C}))^0$ by its flat subscheme (cf Lemma 2 p. 29) and obtain the total complex for a filtered perfect complex. We call this the *perfect subfiltration* denoted $W^{\mathrm{fl}} \subset W$.

Annihilator ideals for filtered hemiperfect complexes

In the spirit of the previous remark, we can also look at the annihilator ideals of a hemiperfect complex. This discussion constitutes, of course, an abelian version of the previous discussion of annihilator ideals for n-stacks, and could have been done before as a motivation for the nonabelian version of the discussion. As the notation is actually simpler in the general nonabelian case, we have not chosen that order of things.

Suppose the base B is $\mathbf{A}^1$ or $\mathcal{A}$. If $C^{\cdot}$ is a hemiperfect complex, then we define as before the *annihilator ideals* $Ann(t^m; C^{\cdot})$ which are ideals on $H^0(C^{\cdot})$. In this case they are ideals of linear subspaces (in particular they are contained in the maximal ideal defining the origin, in other words condition **A2** is always satisfied). To define the annihilator ideals, $C^0 \to C^{\cdot}$ serves as

a smooth surjection from a scheme, and the annihilator ideals are defined as descending from the corresponding ideals defined on the vector scheme C^0. In particular, if $C^{\cdot}$ were actually a perfect complex (i.e. if C^0 were a vector bundle) then the annihilator ideals would vanish.

It also follows that the annihilator ideals are compatible with the Dold-Puppe construction.

If the base is $\mathcal{A}$ so in fact $C^{\cdot}$ corresponds to a filtered complex $(\mathcal{C}, W)$ (which we call a "truncated filtered complex") then we denote the annihilator ideals by $Ann^W(t^m; \mathcal{C})$.

We have the following relationship between the annihilator ideals and the spectral sequence of a filtered complex. It generalizes Lemma 5 one step further, to spectral sequences degenerating at E_2.

Lemma 6 *Suppose $(\mathcal{D}, W)$ is a filtered perfect complex of length $n + k$, and let $\mathcal{C} = \Omega^k \mathcal{D}$ be the loop complex which is a filtered hemiperfect complex of length n. Suppose that the spectral sequence $E_{\cdot}(\mathcal{D})$ degenerates at E_2. Then we have condition* **A1** *saying that*

$$\forall m \geq 1, \quad Ann^W(t^m; \mathcal{C}) = Ann^W(t; \mathcal{C}).$$

Proof: Write $\mathcal{D} = \{D^i, \ d(i) : D^i \to D^{i+1}\}$ with D^i being filtered vector spaces, which we can think of as being vector bundles over $\mathcal{A}$. We obtain complexes $\{E_r^i, \ d_r(i) : E_r^i \to E_r^{i+1}\}$ forming the spectral sequence.

Fix i; then the behavior of the differentials $d_r(i)$ for a given i (i.e. their vanishing or not) depends only on the original differential between filtered vector spaces $d(i)$. To see this, note that if we consider the complex $\tilde{D}^{\cdot}$ consisting only of the terms D^i and D^{i+1} then this gives rise to a spectral sequence $\tilde{E}_r^{\cdot}$ and

$$\tilde{E}_r^i \to E_r^i$$

is surjective, and

$$E_r^{i+1} \to \tilde{E}_r^{i+1}$$

is injective, and the differential $\tilde{d}_r(i)$ factors through $d_r(i)$; thus $d_r(i) = 0$ if and only if $\tilde{d}_r(i) = 0$.

We claim that if $d_r(0) = 0$ for all $r \geq 2$, then the condition of the lemma holds for the truncated complex $\mathcal{C}$. Note that the condition of the lemma, for the truncated complex $\mathcal{C}$, again depends only on the differential $d(0) : D^0 \to D^1$ of the original complex. In view of all of this, we can reduce

to consideration of a complex $\mathcal{D}$ concentrated only in degrees 0 and 1; for the rest of the proof we make this hypothesis.

Now, our complex $\mathcal{D}$ breaks up into a direct sum of complexes where either one of the terms is zero, plus a sum of complexes of the form

$$d : L \to M$$

where L and M are one-dimensional filtered vector spaces and d is a nonzero map between them. To see this decomposition, work with

$$d(0) : D^0 \to D^1$$

as a morphism of vector bundles over $\mathcal{A}$ and use the Gauss method to reduce the matrix of $d(0)$ to a diagonal form (taking note of the $\mathbf{G}_m$-equivariance, which is preserved if the Gauss operations are done correctly).

Now in view of this decomposition (and noting that the cases where $D^0 = 0$ or $D^1 = 0$ are trivial), it suffices to treat the case of a complex of the form $d : L \to M$ with L and M being one-dimensional. Let l and m denote the respective weights of L and M (i.e. the places where the single nonzero steps occur in the filtrations). Note that $m \leq l$. In this case, it is easy to see that the condition of degeneration at E_2 is equivalent to the condition $m \geq l - 1$. From this we deduce that the differential, considered as a map between trivial bundles over $\mathbf{A}^1$, is multiplication either by 1 or by t (here, pull back from $\mathcal{A}$ to $\mathbf{A}^1$). From this, we get the desired statement about the annihilator ideals. |||

A decomposition for $\mathbf{G}_m$-equivariant perfect complexes

In a Dold-Puppe linearized filtration, the object $LGr^W(V)$ is a $\mathbf{G}_m$-equivariant hemiperfect complex. In this subsection we point out that such a thing decomposes into terms $LGr_k^W(V)$ according to the character of the action of $\mathbf{G}_m$.

The choice of sign is motivated by the discussion of filtered vector spaces on p. 17.

Proposition 7 *Let X be any n-stack. A hemiperfect complex C over $X \times B\mathbf{G}_m$, breaks up canonically as a direct sum*

$$C = \bigoplus_k C_k$$

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where the C_k are hemiperfect complexes on $X \times B\mathbf{G}_m$ such that the action of $\mathbf{G}_m$ on the cohomology vector sheaves of C_k is via the character $t \mapsto t^{-k}$.

Sketch of proof: The proof uses two techniques which we don't develop here (but which will, one hopes, be developped elsewhere). These techniques are: tensoring hemiperfect complexes by line bundles; and direct images of hemiperfect complexes. Let ω^k be the rank one line bundle on $B\mathbf{G}_m$ corresponding to the character $t \mapsto t^k$ (and use the same notation for its pullback to $X \times B\mathbf{G}_m$). Denoting by $p : X \times B\mathbf{G}_m \to X$ the projection, put

$$C_k := \omega^{-k} \otimes p^*(p_*(C \otimes \omega^k))$$

where p_* is the (full derived) direct image functor for hemiperfect complexes, which in this case gives a hemiperfect complex as an answer. We have a morphism of hemiperfect complexes

$$i : C \to \bigoplus C_k.$$

Noting that the higher cohomology of $B\mathbf{G}_m$ with vector sheaf coefficients vanishes, and that the global sections functor just takes the invariants, we obtain that i is an isomorphism and that the C_k have the desired property.
|||

If the complex C is perfect then the components C_k are also perfect. This is because the fact that the higher cohomology of $B\mathbf{G}_m$ vanishes means that no truncation operation is necessary to give back complexes supported in $[-n, 0]$, and the direct image of a perfect complex remains perfect.

Remark: The construction of the C_k and the isomorphism i, are $n + 1$-functorial in C and in X. In particular we obtain a morphism of $n + 1$-stacks

$$\underline{Hom}(B\mathbf{G}_m, nHPERF) \to \prod_k nHPERF,$$

$$C \mapsto \left(\ldots, p_*(C \otimes \omega^{-k}), \ldots \right)$$

which is fully faithful with essential image the "restricted product" (consisting of those elements which are zero except for a finite number of places).

Applying the theorem to the case $X = \mathcal{A}$, we obtain:

Corollary 8 *If (C, F) is a $\mathbf{G}_m$-equivariant filtered perfect or hemiperfect complex then it decomposes canonically as a direct sum of terms (C_k, F) such that the action of $\mathbf{G}_m$ on the $H^i(Tot^F(C_k))$ is via the character $t \mapsto t^{-k}$.*

Notation: If $(V, W, LGr^W(V))$ is a Dold-Puppe linearized filtration then we denote by $LGr_k^W(V)$ the component of degree k in the decomposition given by the previous corollary (recall that $LGr^W(V)$ is a $\mathbf{G}_m$-equivariant perfect complex). Similarly, if (V, W) is a filtered (hemi)perfect complex then $Gr^W(V)$ is again a $\mathbf{G}_m$-equivariant (hemi)perfect complex, and again the previous corollary gives a decomposition into components denoted $Gr_k^W(V)$. We call these components (in both cases) the "k-th graded pieces" of the associated-gradeds.

Exercise: The non-canonical decomposition of p. 23 also holds for $\mathbf{G}_m$-equivariant perfect complexes; in other words, if $C : B\mathbf{G}_m \to nPERF$ is a $\mathbf{G}_m$-equivariant perfect complex then it decomposes as

$$C = \bigoplus_i H^{-i}(C)[i],$$

non-canonically but still into $\mathbf{G}_m$-equivariant pieces.

Weight-filtered n-stacks

Before getting to the full notion of nonabelian mixed Hodge structure, we will first discuss just that part which concerns the weight filtration. Note that this is what is new in the present paper: the Hodge filtration has been the subject of several previous papers, and its integration into the picture is essentially straigthforward. On the other hand, for the weight filtration, we have introduced an essentially new idea which is that of "linearization" cf page 25. To explain this alone we introduce the definition of a "weight-filtered n-stack".

Definition: A *pre-weight-filtered n-stack* is a quadruple

$$\{(V, W), \ LGr^W(V), \ \zeta\}$$

where (V, W) is a filtered geometric n-stack (cf page 18 and page 26), where $LGr^W(V)$ is a hemiperfect complex of length n (cf page 23), and

$$\zeta : DP(LGr^W(V)) \cong Gr^W(V)$$

is a Dold-Puppe linearization (cf page 25).

In other words, the $n + 1$-stack of pre-weight filtered n-stacks is just $F^{DP}.nGEOM$.

Recall that we have considered two conditions which one can put on a filtration:

—$\mathbf{A1}(V, W)$: that the annihilator ideals $Ann^W(t^m; V)$ are all equal for $m \geq 1$; qnd

—$\mathbf{A2}(V, W)$: that the annihilator ideal $Ann^W(t; V)$ is contained in the ideal of the zero section of $Gr^W(Tot^F(V))$.

A *weight-filtered n-stack* is a pre-weight-filtered n-stack which satisfies conditions $\mathbf{A1}(V, W)$ and $\mathbf{A2}(V, W)$.

An example: deformation to the tangent space

An important basic example of a pre-weight-filtered n-stack is the following, which puts a linearized filtration on any pointed geometric n-stack. This discussion is closely related to Fulton's "deformation to the normal cone" see [19] [20] [50].

Let

$$p' : Q' \to \mathbf{A}^1$$

be the closed subscheme of $\mathbf{A}^2$ defined by the equation $(t-u)(t+u) = 0$ with projection p' given by the coordinate t. Let $Q := Q'/\mathbf{G}_m$ where the action of $\mathbf{G}_m$ on Q' is by homotheties. This action covers the standard action on $\mathbf{A}^1$ so p' induces a projection

$$p : Q \to \mathcal{A}.$$

Let $P' = \mathbf{A}^1 \subset Q'$ be the subspace defined by $(t - u) = 0$ (it defines a section of p') and let $P = P'/\mathbf{G}_m$. Thus P is the image of a section of p (i.e. $P = \mathcal{A}$). The projection $p : Q \to \mathcal{A}$ is flat, projective and the fibers are finite schemes. The fiber over $[1]$ is a scheme with two points (and $P_{[1]}$ is one of the two points) whereas the fiber over $[0]$ is the spectrum of the dual numbers $\mathbf{C}[u]/u^2$ with $P_{[0]}$ being the basepoint and the $\mathbf{G}_m$-action being by homotheties.

If Y is a geometric n-stack then

$$\underline{Hom}\left(\frac{Q}{\mathcal{A}}, \frac{Y \times \mathcal{A}}{\mathcal{A}}\right)$$

is a geometric n-stack mapping to $\mathcal{A}$. To see this (which is an easy case of (Conjecture 1, [62], p. 27) or alternatively a relative version of (Theorem 8.1, *op cit*)), note that the construction of taking relative internal $\underline{Hom}$ from Q to an object, is compatible with homotopy fiber products and takes smooth

morphisms to smooth morphisms. In the case of Y being a smooth scheme, it is clear that the $\underline{Hom}$ stack defined above is again a smooth scheme over $\mathcal{A}$. Now any scheme is obtained from smooth ones as a homotopy fiber product, so we get the desired statement for Y a scheme; finally, it follows for any geometric n-stack Y by applying the definition of geometric n-stack, which only involves notions of smoothness, of being a scheme, and of homotopy fiber products.

The operation of restricting a morphism to the section P provides a morphism

$$\underline{Hom}(\frac{Q}{\mathcal{A}}, \frac{Y \times \mathcal{A}}{\mathcal{A}}) \to \underline{Hom}(\frac{P}{\mathcal{A}}, \frac{Y \times \mathcal{A}}{\mathcal{A}}) = Y \times \mathcal{A}.$$

Suppose now that $y : * \to Y$ is a point. Put

$$Tot^{W(y)}(Y) := \underline{Hom}\left(\frac{Q}{\mathcal{A}}, \frac{Y \times \mathcal{A}}{\mathcal{A}}\right) \times_{Y \times \mathcal{A}} \mathcal{A},$$

where the first morphism is the restriction morphism described above and the second morphism is $(y, 1) : \mathcal{A} \to Y \times \mathcal{A}$. This is a geometric n-stack mapping to $\mathcal{A}$. From the previous descriptions of the fibers of Q over [1] and [0] respectively, we get immediately that the fiber of $Tot^{W(y)}(Y)$ over [1] is Y and that the fiber over [0] is the tangent stack $T_y(Y)$ (see [62] for the definition of the tangent stack). Thus, by definition $Tot^{W(y)}(Y)$ defines a filtration $W(y)$ of Y, whose associated-graded is the tangent stack to Y at y.

Furthermore, recall from [62] that the tangent stack $T_y(Y)$ actually has a structure of "spectrum", i.e. it has a canonical infinite delooping structure. Since we are in characteristic zero, a spectrum is the same thing as a complex, thus $T_y(Y)$ has a structure of perfect complex (one can check in the construction that this structure is equivariant for the action of $\mathbf{G}_m$ by homotheties on $Spec(\mathbf{C}[u]/u^2)$). Letting $LGr^{W(y)}(Y)$ denote this perfect complex whose Dold-Puppe is $Gr^{W(y)}(Y)$, we obtain a linearization of the filtration $W(y)$, which makes $(Y, W(y), LGr^{W(y)}(Y))$ into a pre-weight-filtered geometric n-stack.

We obtain an $n + 1$-functor

$$nGEOM^{ptd} \to F^{DP}.nGEOM.$$

Exercises: (1) Observe that condition **A2** is always satisfied (indeed $Tot^{W(y)}(Y)$ comes with a section.

(2) Show that if Y is a scheme with quadratic singularities at y (or if Y is a geometric n-stack with a smooth surjection from such a scheme) then $W(y)$ satisfies condition **A1**. The subscheme defined by the annihilator ideal is the normal cone.

While we don't imagine that all weight filtrations arise in this way, we do think that this should provide an important construction of certain types of weight filtrations. The relationship between weight filtrations arising "in nature" and those arising from this construction, seems to be an interesting question for further thought.

Interaction with the Hodge filtration

We apply the discussion from page 21 to the case of the weight filtration. A *weight-Hodge linearized bifiltration* of a geometric n-stack is an object of $F.LF.nGEOM = F.F^{DP}.nGEOM$ where $DP : nHPERF \rightarrow nGEOM$ is the Dold-Puppe linearization functor considered above. This object is denoted (V, W, F) where V is a geometric n-stack; where (V, W) is the "weight-filtration" on V which is the one which is linearized, i.e. it is a filtration plus a ($\mathbf{G}_m$-equivariant) perfect complex $LGr^W(V)$ whose associated ($\mathbf{G}_m$-equivariant) geometric n-stack is $Gr^W(V)$; and finally where F is a filtration of (V, W) in the stack $F^{DP}.nGEOM$. As described above, concretely this means that $Tot^F(V, W)$ is a Dold-Puppe-linearized filtered geometric n-stack over $\mathcal{A}$. In turn this means that there is a $\mathbf{G}_m$-equivariant filtered perfect complex

$$(LGr^W(V), F) : B\mathbf{G}_m \rightarrow F.nHPERF$$

whose underlying filtered geometric n-stack (Dold-Puppe) is $(Gr^W(V), F)$.

In order to clarify what is going on, when we look at both the weight filtration and the Hodge filtration we will use two different subscripts for the two copies of $\mathcal{A}$ which come into the picture: the copy which concerns the weight filtration will be denoted by $\mathcal{A}_{wt}$ and the copy concerning the Hodge filtration will be denoted by $\mathcal{A}_{hod}$. Thus a weight-Hodge linearized bifiltration is a morphism of geometric n-stacks

$$Tot^{W,F}(V) \rightarrow \mathcal{A}_{hod} \times \mathcal{A}_{wt},$$

together with a perfect complex $Tot^F(LGr^W(V))$ over $[0]_{wt} \times \mathcal{A}_{hod}$.

In the presence of both weight and Hodge filtrations, we would like to impose our two conditions **A1** and **A2** on the weight filtration, plus some

additional conditions related to the Hodge filtration. These conditions are quite naturally imposed on the biggest stack for which they make sense, in the following way.

—$\mathbf{A1}(V, W, F)$: that the annihilator ideals $Ann^W(t^m; Tot^F(V))$ satisfy the condition $\mathbf{A1}(Tot^F(V), W)$ (they are all equal for $m \geq 1$);

—$\mathbf{A2}(V, W, F)$: that the annihilator ideal $Ann^W(t; Tot^F(V))$ satisfies condition $\mathbf{A2}(Tot^F(V), W)$, namely that it is contained in the ideal of the zero section of $Gr^W(Tot^F(V))$; and

—$\mathbf{Fl}(V, W, F)$: that the Hodge filtration F is flat, by which we mean that the morphism corresponding to the Hodge filtration,

$$Tot^{W,F}(V) \to \mathcal{A}_{hod}$$

is flat.

These three conditions are meant to insure that the biflitration (V, W, F) is reasonably well connected to V itself. Condition $\mathbf{A2}$ prevents silly constructions such as in Counterexample (1) of p. 32. The flatness condition means that the only degeneracies involved are the non-flatness of the weight filtration. Note in particular that because of the flatness condition, $Tot^F Gr^W(V) \to \mathcal{A}_{hod}$ is flat, which implies that the hemiperfect linearization is actually a perfect one, i.e. $Tot^F Gr^W(V)$ is a perfect complex. Condition $\mathbf{A1}$ puts a limit on how complicated the degeneracy in the weight-filtration direction can be.

To bring things back down to earth a bit, we take note of the following remarks which follow immediately from the definitions. The homotopy stacks $\pi_i(Gr^W(V))$ are vector sheaves over $[0]_{wt} \cong B\mathbf{G}_m$, i.e. they are finite dimensional complex vetor spaces with $\mathbf{G}_m$-action. This includes the cases $i = 0$ and $i = 1$ because they are the cohomology sheaves of the perfect complex $LGr^W(V)$:

$$\pi_i(Gr^W(V)) = H^{-i}(LGr^W(V)).$$

In the case $i = 0$ this means that $\pi_0(Gr^W(V))$ is given the extra structure of a complex vector space with $\mathbf{G}_m$-action. Furthermore, a perfect complex over $B\mathbf{G}_m$ splits up (non-canonically) as the product of its cohomology sheaves, so $Gr^W(V)$ splits as a product of its Postnikov-Eilenberg-MacLane components (cf the exercise on p. 38):

$$Gr^W(V) \cong \prod_{i=0}^{n} K(\pi_i(Gr^W(V))/B\mathbf{G}_m, i).$$

Now the Hodge filtration induces a filtration on $LGr^W(V)$. This means that $(LGr^W(V), F)$ corresponds to a perfect complex

$$Tot^F(LGr^W(V)) \to \mathcal{A}_{hod}.$$

In particular the cohomology sheaves of this are vector sheaves on $\mathcal{A}_{hod}$, so they correspond to filtrations of the vector spaces $\pi_i(Gr^W(V))$ in the sense that these latter are objects of $F.VESH$ (cf p. 17). This leads us to our next condition, stating that these vector sheaves are actually vector bundles (and thus that the filtration is a filtration in the classical sense i.e. an object of $F.VECT$). We call this condition "strictness" (see Lemma 5):

—$\mathbf{Str}(V, W, F)$: that the cohomology sheaves

$$H^{-i}(Tot^F(LGr^W(V)))$$

over $\mathcal{A}_{hod}$, are actually vector bundles.

We can view $(LGr^W(V), F)$ as a filtered complex, and by Lemma 5 this condition amounts to asking that it be a *strict* filtered complex in the sense of [13] [14]. This condition means, in view of the discussion of $\mathbf{G}_m$-equivariant perfect complexes, that $Tot^F(LGr^W(V))$ (which is a $\mathbf{G}_m$-equivariant perfect complex over $\mathcal{A}_{hod}$) decomposes as a direct sum of its cohomology vector bundles. Recall from Lemma 5 that it is equivalent to degeneration of the spectral sequence for the Hodge filtration F on $LGr^W(V)$, at the E_1-term.

In view of the equivalence $Gr^W(V) = \Phi(LGr^W(V))$ the above strictness condition can be rewritten as saying that the $\pi_i(Tot^F(Gr^W(V)))$ are vector bundles over $\mathcal{A}$.

Analytic and real structures

In the main definitions to come in Part II, we shall need some basic definitions about analytic objects and real structures, particularly as they relate to filtered objects.

Algebraic to analytic objects

Let $\mathcal{Z}^{an}$ denote the site of paracompact complex analytic spaces with the etale topology. We have a functor $\Phi : \mathcal{Z} \to \mathcal{Z}^{an}$ defined by $\Phi(X) := X^{an}$. Define the operation $\mathcal{F} \mapsto \mathcal{F}^{an}$ from n-stacks on $\mathcal{Z}$ to n-stacks on $\mathcal{Z}^{an}$, by the formula

$$\mathcal{F}^{an} := \Phi_!(\mathcal{F}),$$

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where we put (from [31])

$$\Phi_{!,pre}(\mathcal{F})(Y) := \operatorname{colim}_{Y \to \Phi(X)} \mathcal{F}(X),$$

and we define $\Phi_!(\mathcal{F})$ to be the n-stack associated to the n-prestack $\Phi_{!,pre}(\mathcal{F})$.

The *colim* in the definition or $\Phi_{!,pre}$ is of course the homotopy colimit of n-stacks (and since we're talking about n-stacks of groupoids, this is the same as the homotopy colimit of the diagram of spaces [6]).

We have the standard notion of smooth morphism between complex analytic spaces, and by the construction of [62] this gives rise to the notion of *analytic geometric n-stack*. Note however that in the analytic situation we don't make any analogue of the "very presentable" condition.

The following lemma gives the fundamental compatibility between the analytic and algebraic notions of geometricity.

Lemma 9 *The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is compatible with homotopy fiber products. Furthermore, it takes a scheme to its associated analytic space. As a consequence, it takes geometric n-stacks to analytic geometric n-stacks.*

Proof: The *hocolim* involved in the definition of $\Phi_{!,pre}$ is filtering (because it is taken over a category which admits finite products and fiber products), therefore it commutes with finite homotopy limits. The operation of taking the associated n-stack is also compatible with finite homotopy limits, see [31] [51], so the operation $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is compatible with finite homotopy limits. The second sentence is standard; and the third sentence is an immediate corollary of the fact the definition of "geometric n-stack" (resp. "analytic geometric n-stack") refers only to homotopy fiber products, schemes (resp. analytic spaces) and smoothness of maps between schemes (resp. smoothness of maps between analytic spaces). |||

To understand in a simple way what is going on in the above lemma, suppose $\mathcal{F}$ is a geometric n-stack with a surjective morphism from a scheme $Z \to \mathcal{F}$. Let $R = Z \times_{\mathcal{F}} Z$, which is a geometric $(n-1)$-stack. Thus in a certain sense $\mathcal{F}$ is presented as the quotient of Z by the "relation" R. Now $Z^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ is a smooth surjection and

$$R^{\mathrm{an}} = Z^{\mathrm{an}} \times_{\mathcal{F}^{\mathrm{an}}} Z^{\mathrm{an}}.$$

Thus $\mathcal{F}^{\mathrm{an}}$ is presented as the quotient of Z^{an} by R^{an}.

We define filtrations in the same way as in the algebraic case, using the analytic stack
$$\mathcal{A}^{\mathrm{an}} := (\mathbf{A}^1)^{\mathrm{an}}/(\mathbf{G}_m)^{\mathrm{an}}.$$
(which is the analytic stack associated to $\mathcal{A}$). Thus for example a filtered analytic geometric n-stack is a morphism of analytic geometric n-stacks $T \to \mathcal{A}^{\mathrm{an}}$. The only construction we need is that if (V, W) is a filtered algebraic geometric n-stack then by taking the analytic n-stack T^{an} associated to the total space $T = Tot^W(V)$ we obtain a filtered analytic geometric n-stack (V^{an}, W).

An *analytic perfect complex* (resp. *analytic hemiperfect complex*) over a base B is a morphism $B \to nPERF^{\mathrm{an}}$ (resp. $B \to nHPERF^{\mathrm{an}}$). If B is a complex analytic space, this is the same thing as a complex of sheaves of $\mathcal{O}^{\mathrm{an}}$-modules on B which is locally quasiisomorphic to a complex of vector bundles supported in $[-n, 0]$ (resp. the same but with C^0 a vector scheme [5]). Again, given an algebraic perfect complex over a base X we obtain an associated analytic perfect complex over X^{an}.

An important observation in what follows is that an analytic perfect complex over $\mathcal{A}^{\mathrm{an}}$ has a canonical algebraic structure (because of the $\mathbf{G}_m$-action on $\mathbf{A}^1$ whose orbit goes out to the only point at infinity). In particular, if (U_1, W) and (U_2, W) are algebraic filtered perfect complexes, then any morphism between the associated analytic filtered complexes, is actually algebraic. This remark will be used in defining the notion of linearization of a pre-namhs: whereas there is an analytic equivalence involved in the notion of pre-namhs, for the "linear" object this equivalence is taken to be algebraic because no greater generality would be obtained by using an analytic equivalence.

Real structures

We treat here the extension of ground fields $[\mathbf{C} : \mathbf{R}]$. Most of what we say should be valid for an arbitrary extension of ground field (although in

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[5] We define an analytic vector scheme as being something which is locally the kernel of a map of finite rank vector bundles. Now the 1-stack of analytic vector schemes is the analytic stack associated to $VESCH$ because any analytic vector scheme is pulled back from a sort of "universal" algebraic vector scheme over the space of matrices, by an analytic map. One conjectures that this definition of analytic vector scheme is equivalent to the notion of "vector space in the category of complex analytic spaces over the base" but we don't have a proof of that.

the case of infinite field extensions, some extra work is necessary because we have to go out of the realm of schemes of finite type). In fact we state many things without proof, things which would be more appropriately dealt with in the general context of an arbitrary field extension.

Let $\mathcal{Z}_{\mathbf{R}}$ denote the category of schemes of finite type over $\mathbf{R}$ with the etale topology. We have a complexification functor $\Psi : \mathcal{Z}_{\mathbf{R}} \to \mathcal{Z}$ which sends a real scheme to its associated complex scheme. On the level of schemes, this functor is defined by

$$\Psi(X) := X \times_{Spec(\mathbf{R})} Spec(\mathbf{C}).$$

As in the previous section this gives a functor $\Psi_!$ from n-stacks on $\mathcal{Z}_{\mathbf{R}}$ to n-stacks on $\mathcal{Z}$. We denote this operation by $X \mapsto X_{\mathbf{C}}$; often an object on $\mathcal{Z}_{\mathbf{R}}$ will be denoted by $X_{\mathbf{R}}$ in which case the complexified object is denoted $X_{\mathbf{C}}$.

The formula for $\Psi_!$ is

$$\Psi_!(X)(Y) := \operatorname{colim}_{Y \to \Psi(U)} X(U),$$

where X is a stack on $\mathcal{Z}_{\mathbf{R}}$ and $Y \in \mathcal{Z}$ is a complex scheme. The colimit is taken over real schemes U with maps $Y \to \Psi(U)$. Note in this case however (this is a difference with the analytification functor) that the category over which the colimit is taken has a final object. Indeed Ψ acting on schemes has a left adjoint $\Psi^!$ defined by putting $\Psi^!(Y)$ equal to the real scheme obtained by composing $Y \to Spec(\mathbf{C}) \to Spec(\mathbf{R})$. A morphism of complex schemes

$$Y \to \Psi(U) = U \times_{Spec(\mathbf{R})} Spec(\mathbf{C})$$

is the same thing as a morphism of real schemes $\Psi^!(Y) \to U$ (take the first projection of the previous). Thus the colimit may be seen as a colimit over maps $\Psi^!(Y) \to U$ and we obtain the formula

$$\Psi_!(X)(Y) := X(\Psi^!(Y)).$$

On stacks the functor $\Psi_!$ has a left adjoint again denoted $\Psi^!$ which, heuristically, takes a complex stack $X \to Spec(\mathbf{C})$ to the real stack obtained by composing $X \to Spec(\mathbf{C}) \to Spec(\mathbf{R})$. (Technically speaking this definition doesn't make sense because X is a stack over the site $Sch/\mathbf{C}$ and $Spec(\mathbf{R})$ doesn't exist in that context; to be correct, $\Psi^!$ must be defined by the adjoint property.) There is a natural transformation $\Psi^!(X) \to \Psi^! Spec(\mathbf{C})$ and this

gives an equivalence of $n+1$-categories between the n-stacks over $\mathcal{Z}$ and the n-stacks over $\mathcal{Z}_{\mathbf{R}}$ provided with morphism to $\Psi^! Spec(\mathbf{C})$. Using this point of view (plus a slight abuse of notation) it makes sense to write

$$\Psi_!(X) = X \times_{Spec(\mathbf{R})} Spec(\mathbf{C}).$$

If X is an n-stack on $\mathcal{Z}$ then we say that a *real structure* on X is the specification of a real n-stack $Y_{\mathbf{R}}$ plus an equivalence $X \cong Y_{\mathbf{C}}$.

There is of course a "Galois" version of this notion, namely if X is a complex stack then composing with the complex conjugation involution $\mathcal{Z} \to \mathcal{Z}$ we obtain a new complex stack denoted $\overline{X}$; and a real structure on X is the same thing as a twisted $\mathbf{Z}/2$-action on X consisting principally of an equivalence $X \cong \overline{X}$ plus some higher coherence data. We don't wish to go into further detail on this at present. Note however that in terms of the "Galois" point of view, $\Psi^!(X)$ is the scheme $X \coprod \overline{X}$ with real structure induced by the involution which exchanges the two pieces.

As usual there is a notion of smooth map between real schemes, and using this notion we obtain following the procedure of [62], the notion of *real geometric n-stack* (which is an n-stack on $\mathcal{Z}_{\mathbf{R}}$). We impose our tacit hypothesis of the introduction, that included in the condition of being a real geometric n-stack is the condition that the associated complex geometric n-stack is also very presentable. As the condition is being made on the complexification, it concerns the homotopy group sheaves for all choices of basepoint in the complexified stack. Nonetheless, one can say that for real basepoints the homotopy group sheaves of the complex stack are the complexifications of those of the real stack. In the connected case when there is only one choice of basepoint up to equivalence, this determines the homotopy group sheaves of the complexified stack.

Lemma 10 *The operation of complexification of a real n-stack is compatible with homotopy fiber products. Therefore it sends real geometric n-stacks to geometric n-stacks. Furthermore, if $X_{\mathbf{R}}$ is an n-stack on $\mathcal{Z}_{\mathbf{R}}$ then $X_{\mathbf{R}}$ is a real geometric n-stack if and only if its complexification $X_{\mathbf{C}}$ is a geometric n-stack.*

Proof: From the formula $\Psi_!(X)(Y) = X(\Psi^!(Y))$ we obtain immediately the compatibility with homotopy fiber products. This yields compatibility with the notion of geometricity (compatibility with the very presentable condition is tautological since we impose that condition on the complexified stack to

begin with). To prove the last sentence, suppose $X_{\mathbf{R}}$ is a real stack and suppose $Y \to X_{\mathbf{C}}$ is a smooth surjection. Then

$$Y \times_{X_{\mathbf{C}}} \overline{Y} \to X_{\mathbf{C}}$$

is a smooth surjection which admits a definition over $\mathbf{R}$ (the involution exchanges the two factors Y and $\overline{Y}$). Using this we can prove that if $X_{\mathbf{C}}$ is geometric then so is $X_{\mathbf{R}}$. |||

Real structures on moduli stacks

Let $nGEOM_{\mathbf{R}}$ denote the stack of real geometric n-stacks over $\mathcal{Z}_{\mathbf{R}}$, defined by setting $nGEOM_{\mathbf{R}}(U)$ equal to the $n + 1$-category of real geometric n-stacks $X \to U$.

There is also a notion of real perfect (resp. hemiperfect) complex, which just means a perfect (resp. hemiperfect) complex of real vector bundles on the real scheme. This gives an $n+1$-stack $nPERF_{\mathbf{R}}$ (resp. $nHPERF_{\mathbf{R}}$) over $\mathcal{Z}_{\mathbf{R}}$. A perfect complex on a real n-stack $X_{\mathbf{R}}$ is by definition a morphism $X_{\mathbf{R}} \to nPERF_{\mathbf{R}}$ (resp. $X_{\mathbf{R}} \to nHPERF_{\mathbf{R}}$).

Theorem 1 *We have*

$$(nGEOM_{\mathbf{R}})_{\mathbf{C}} = nGEOM \quad and \quad (nHPERF_{\mathbf{R}})_{\mathbf{C}} = nHPERF.$$

In other words, the complex moduli stacks $nGEOM$ and $nHPERF$ have canonical real structures.

Proof: Suppose Y is a complex scheme. Then $(nGEOM_{\mathbf{R}})_{\mathbf{C}}(Y)$ is by definition the $n + 1$-category of real geometric n-stacks over $\Psi^!(Y)$. These are thus stacks X fitting into a diagram

$$X \to \Psi^! Y \to \Psi^! Spec(\mathbf{C}) \to Spec(\mathbf{R}).$$

Looking only at the first three terms of this diagram and applying the discussion above which gives an equivalence between n-stacks over $\mathcal{Z}$ and n-stacks over $\mathcal{Z}_{\mathbf{R}}$ provided with morphism to $\Psi^! Spec(\mathbf{C})$, we obtain that the data of X is the same as the data of an object in $nGEOM(Y)$. This gives the equivalence in question for $nGEOM$. For $nHPERF$ we can deduce it directly by considering a hemiperfect complex as an $n + N$-stack which is N-connected, for $N \geq n + 1$, whose Ω^N is geometric. |||

Real structures and filtrations

Now we turn to filtrations. The stack $\mathcal{A} := \mathbf{A}^1/\mathbf{G}_m$ is defined over $\mathbf{R}$, in other words we have a real n-stack

$$\mathcal{A}_{\mathbf{R}} := \mathbf{A}^1_{\mathbf{R}}/\mathbf{G}_{m,\mathbf{R}}.$$

A *real filtration of a real geometric n-stack* $V_{\mathbf{R}}$, denoted $(V_{\mathbf{R}}, W)$, is a real geometric n-stack $Tot^W(V_{\mathbf{R}})$ over $\mathcal{A}_{\mathbf{R}}$. The associated complex n-stack is thus a filtration of $V_{\mathbf{C}}$, we denote this associated filtration by $(V_{\mathbf{C}}, W)$, with its total space

$$Tot^W(V_{\mathbf{C}}) := Tot^W(V_{\mathbf{R}})_{\mathbf{C}}.$$

The associated-graded $Gr^W(V_{\mathbf{R}})$ is a real geometric n-stack with action of $\mathbf{G}_{m,\mathbf{R}}$. Again, this complexifies to $Gr^W(V_{\mathbf{C}})$.

We take note of the fact that the canonical decomposition of $\mathbf{G}_m$-equivariant perfect complexes (see pages 36 or 37) is compatible with the real structure.

A weight-filtered n-stack with real structure (V, W) means a collection of real stacks just as in the definition of page 25. In this case, the $\pi_i Gr^W(V) \cong H^{-i}(LGr^W(V))$ are real vector spaces with action of $\mathbf{G}_{m,\mathbf{R}}$. The components $H^{-i}(LGr^W_k(V))$ are thus real subspaces. In the next part of the paper, this will be combined with a "Hodge filtration" which induces a complex filtration F on the vector space $H^{-i}(LGr^W(V))_{\mathbf{C}}$, i.e. an actual filtration by subspaces $F^p H^{-i}(LGr^W(V))_{\mathbf{C}}$. It thus makes sense to take the complex conjugate of this filtration with respect to the real structure, giving a filtration by subspaces $\overline{F}^q H^{-i}(LGr^W(V))$. This allows us to impose the condition that the Hodge filtration and its complex-conjugate be $k-i$-opposed. This is the only place where the real structures which will be involved in the definitions to follow, are actually used.

Part II: The main definitions and conjectures

We now present the main definitions of pre-nonabelian mixed Hodge structure and nonabelian mixed Hodge structure, as well as a number of the basic conjectures which serve, we hope, to explain our motivation for making these definitions.

One small point to notice is that the notion of pre-namhs is not quite analogous to the notion of pre-weight-filtered n-stack which we discussed above. For pedagogical reasons, we included the notion of linearization in the notion of pre-weight-filtered n-stack. However, for practical reasons the notion of pre-namhs, without linearization, seems to be the most useful intermediate version of this definition.

Pre-nonabelian mixed Hodge structures

We can now give our first main definition. A *pre-nonabelian mixed Hodge structure* or *pre-namhs* is a collection

$$\mathcal{V} = \{(V_{DR}, W, F); \ (V_{B,\mathbf{R}}, W); \ \zeta_{\mathcal{V}}\}$$

where:
V_{DR} is a geometric (hence by our convention, very presentable) n-stack;
(V_{DR}, W, F) is a geometric bifiltration of V_{DR} with the filtrations parametrized by $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ (but note that the weight filtration is *not* linearized);
$V_{B,\mathbf{R}}$ is a real geometric n-stack;
$(V_{B,\mathbf{R}}, W)$ is a geometric filtration on $V_{B,\mathbf{R}}$; and
$\zeta_{\mathcal{V}}$ is an equivalence between the associated analytic filtered objects

$$\zeta_{\mathcal{V}} : (V_{DR}, W)^{\text{an}} \cong (V_{B,\mathbf{C}}, W)^{\text{an}}.$$

We obtain an $n{+}1$-*category of pre-namhs* denoted $nPNAMHS$. Because of the presence of analytic morphisms in the definition, it is not immediately useful to consider an $n + 1$-stack of these objects so we don't. Constructing $nPNAMHS$ is straightforward following the definition of a pre-namhs. For example we have the following diagram of $n + 1$-categories

$$F.F.nGEOM(*) \to F.nGEOM(*)$$
$$\searrow$$
$$F.nGEOM^{[\text{an}]}(*)$$
$$\nearrow$$
$$F.nGEOM_{\mathbf{R}}(*) \to F.nGEOM(*)$$

and $nPNAMHS$ is the limit or homotopy fiber product

$$nPNAMHS := F.F.nGEOM(*) \times_{F.nGEOM^{\mathrm{an}}(*)} F.nGEOM_{\mathbf{R}}(*).$$

Here $F.nGEOM^{[\mathrm{an}]}$ is the analyic moduli $n + 1$-stack of filtered analytic geometric n-stacks (which is different from the analytification of $F.nGEOM$), and $F.nGEOM_{\mathbf{R}}$ is the real moduli $n + 1$-stack of filtered real geometric n-stacks. The top horizontal morphism is "forgetting the Hodge filtration" and the bottom horizontal morphism is complexification; the diagonal morphisms are analytification.

Concretely a 1-morphism

$$\varphi : \{(V_{DR}, W, F), (V_{B,\mathbf{R}}, W), \zeta_{\mathcal{V}}\} \rightarrow \{(V'_{DR}, W, F), (V'_{B,\mathbf{R}}, W), \zeta_{\mathcal{V}'}\}$$

in $nPNAMHS$ consists of a morphism of bifiltered objects,

$$\varphi_{Hod} : (V_{DR}, W, F) \rightarrow (V'_{DR}, W, F),$$

a morphism of filtered real n-stacks

$$\varphi_{B,\mathbf{R}} : (V_{B,\mathbf{R}}, W) \rightarrow (V'_{B,\mathbf{R}}, W),$$

and a homotopy or 2-morphism (invertible up to equivalence) between

$$(\varphi_{B,\mathbf{C}})^{\mathrm{an}} \circ \zeta_{\mathcal{V}} \quad \text{and} \quad \zeta_{\mathcal{V}'} \circ (\varphi_{DR})^{\mathrm{an}}.$$

The $n+1$-categories entering into the fiber product defining $nPNAMHS$ admit finite limits, and the morphisms in the fiber product are compatible with these limits; therefore the fiber product $nPNAMHS$ also admits finite limits. In particular we can use homotopy fiber products here. This will not be the case with the notion of "namhs", which explains the utility of isolating out a notion of "pre-namhs".

Pre-mixed Hodge objects

Our definition of pre-namhs could be generalized in the following way. Suppose OBJ is an $n + 1$-stack (not of groupoids though) with real form $OBJ_{\mathbf{R}}$ and with a morphism $OBJ^{\mathrm{an}} \rightarrow OBJ^{[\mathrm{an}]}$ of analytic stacks. Then a *pre-mixed Hodge object in OBJ* is a collection

$$\{(V_{DR}, W, F), \quad (V_{B,\mathbf{R}}, W), \zeta\}$$

where $V_{DR} : * \to OBJ$ is an object of type OBJ; where (W, F) is a bifiltration of V_{DR} i.e.

$$Tot^{W,F}(V_{DR}) : \mathcal{A}_{wt} \times \mathcal{A}_{hod} \to OBJ,$$

with fiber over $[1] \times [1]$ equal to V_{DR}; where $V_{B,\mathbf{R}} : Spec(\mathbf{R}) \to OBJ_{\mathbf{R}}$ is a real object, with filtration W; and where ζ is an equivalence in $OBJ^{[\text{an}]}$ between $Tot^W(V_{DR})$ and the complexification of $Tot^W(V_{B,\mathbf{R}})$. We obtain an $n+1$-category $PMH.OBJ$ of pre-mixed Hodge objects in OBJ (we don't consider an $n+1$-stack of these objects).

A pre-namhs is just a pre-mixed Hodge object in $nGEOM$. Below we will see that a pre-mixed Hodge structure is an object of $PMH.VECT$; a pre-mixed Hodge complex is an object of $PMH.nPERF$; and we shall have occasion to meet objects of $PMH.nHPERF$, $PMH.VESH$, etc.

Split pre-namhs's

Define a *split* pre-namhs to be a $\mathbf{G}_m$-equivariant pre-namhs such that the weight-filtrations arise from the construction of page 18 which to a $\mathbf{G}_m$-equivariant stack associates a $\mathbf{G}_m$-equivariant filtered stack. Denote by $splPNAMHS$ the $n+1$-category of these objects. The operation of "taking the associated-graded for the weight filtration" provides an $n+1$-functor

$$Gr^W : nPNAMHS \to splPNAMHS.$$

Mixed Hodge complexes and linearization

Our next step is the notion of "linearization" for the weight filtrations. For this, we need to go back to the abelian yet higher cohomological case of "mixed Hodge complexes". In order to establish notational conventions we first discuss the notion of mixed Hodge structures. A *pre-mixed Hodge structure* or *pre-mhs* (by which we always mean, in the current paper, pre-$\mathbf{R}$-mhs) is an object of $PMH.VECT$, in other words it is an uple

$$\mathcal{V} = \{(V, W, F); \ (V_{\mathbf{R}}, W), \zeta\}$$

where V is a complex vector space with two filtrations F and W, $V_{\mathbf{R}}$ is a real vector space with real filtration W, and ζ is an isomorphism of filtered objects between (V, W) and the complexification of $(V_{\mathbf{R}}, W)$. In particular, we can think of (V, W, F) as being an object of $F.F.VECT$ and $(V_{\mathbf{R}}, W)$ as an object of $F.VECT_{\mathbf{R}}$.

If $\mathcal{V}$ is a pre-mhs, we obtain the *associated-graded* $Gr^W(\mathcal{V})$ which is a *split* pre-mhs, i.e. it splits up as a direct sum

$$Gr^W(\mathcal{V}) = \bigoplus Gr_k^W(\mathcal{V})$$

of pieces on which the $\mathbf{G}_m$-action is by the character $\lambda \mapsto \lambda^{-k}$. On these pieces, the weight filtrations are one-step (concentrated in degree k) and we have Hodge filtrations and real structures: we can denote these pieces by

$$\{(Gr_k^W(V), F), \ \ Gr_k^W(V)_{\mathbf{R}}\}.$$

In particular, we can take the *complex conjugate* of the Hodge filtration. To do this, use the old-fashioned approach to a filtration and take the complex conjugate of every subspace. According to our conventions (p. 16), F corresponds to an *increasing* filtration $F_p Gr_k^W(V)$, but we can turn it into a *decreasing* filtration by the reindexation

$$F^p Gr_k^W(V) := F_{-p} Gr_k^W(V).$$

The complex conjugate filtration $\overline{F}$ is defined by setting $\overline{F}^p Gr_k^W(V)$ equal to the complex conjugate of $F^p Gr_k^W(V)$ with respect to the real structure $Gr_k^W(V)_{\mathbf{R}}$. Recall from [13] that one says that F and $\overline{F}$ are μ-*opposed on* $Gr_k^W(V)$ if

$$F^p Gr_k^W(V) \cap \overline{F}^q Gr_k^W(V) = 0, \quad p + q > \mu,$$

and

$$Gr_k^W(V) = \bigoplus_{p+q=\mu} F^p Gr_k^W(V) \cap \overline{F}^q Gr_k^W(V).$$

Note that in this condition, we are using the upper-reindexed versions of F. In terms of the lower indexing, we get that F and $\overline{F}$ are μ-opposed on $Gr_k^W(V)$ if

$$F_p Gr_k^W(V) \cap \overline{F}_q Gr_k^W(V) = 0, \quad \mu + p + q < 0,$$

and

$$Gr_k^W(V) = \bigoplus_{p+q+\mu=0} F_p Gr_k^W(V) \cap \overline{F}_q Gr_k^W(V).$$

Here is the place where the conventions are fixed! If s is an integer, we say that a pre-mhs $\mathcal{V}$ is an *s-shifted mhs* if for every k, the filtrations F and

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$\overline{F}$ are $k + s$-opposed on $Gr_k^W(V)$. We say that $\mathcal{V}$ is a *mhs* if it is a 0-shifted mhs.

We now get to mixed Hodge complexes. Fix $n \geq 0$ which will be the length of the interval of support for the mixed Hodge complexes we are considering (we only consider ones of finite length). Define a *pre-mixed Hodge complex* or *pre-mhc* to be an uple

$$\mathcal{C} = \{(C_{DR}, W, F), \ (C_{B,\mathbf{R}}, W), \ \zeta_{\mathcal{C}}\}$$

where C_{DR} is a perfect complex (i.e. a complex of finite dimensional $\mathbf{C}$-vector spaces); W and F form a bifiltration in the category of hemiperfect complexes (see page 23), so on the whole (C_{DR}, W, F) is an object of $F.F.nHPERF$; similarly $(C_{B,\mathbf{R}}, W)$ is a filtered real hemiperfect complex, and $\zeta_{\mathcal{C}}$ is an eqivalence between $(C_{B,\mathbf{C}}, W)$ and (C_{DR}, W) as objects of $F.nHPERF$.

The use of hemiperfect complexes rather than perfect ones is a technical point which the first-time reader can safely ignore. In the definition of "mixed Hodge complex" (i.e. when we take off the prefix "pre-") the hemiperfect complexes will be assumed to be perfect.

Notice here in particular that $\zeta_{\mathcal{C}}$ is an algebraic equivalence—we don't speak of associated analytic objects, because in fact an analytic equivalence would automatically be algebraic, see p. 45.

Let $PMHC$ denote the $n + 1$-category of pre-mixed Hodge complexes. This definition is almost equivalent to the object studied by Deligne in [14], the only difference is our use of hemiperfect rather than perfect complexes, which comes from the fact that we may want to consider truncations of Deligne-type pre-mhc's.

We again have a notion of *split* pre-mhc, which means a $\mathbf{G}_m$-equivariant pre-mhc such that the $\mathbf{G}_m$-action splits the weight filtration. Let $splPMHC$ denote the $n + 1$-category of these objects. As before, there is a functor $\mathcal{C} \mapsto Gr^W(\mathcal{C})$,

$$Gr^W : PMHC \to splPMHC.$$

The canonical decomposition of $\mathbf{G}_m$-equivariant perfect complexes given by Theorem 7 (compatible with filtrations and real structures) gives a decomposition of split pre-mhc: if $\mathcal{C}$ is a split pre-mhc then it decomposes canonically as a direct sum of pre-mhc with trivial (but shifted) weight filtrations $\mathcal{C}_k$. If $\mathcal{C}$ is any pre-mhc then the associated-graded $Gr^W(\mathcal{C})$, as a split-pre-mhc, decomposes as a direct sum which we write

$$Gr^W(\mathcal{C}) = \bigoplus_k Gr_k^W(\mathcal{C}).$$

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Recall that the index k means the piece on which $\mathbf{G}_m$ acts by the character $\lambda \mapsto \lambda^{-k}$.

The Dold-Puppe functor from $nHPERF$ to $nGEOM$ induces in an obvious way functors

$$DP : PMHC \to PNAMHS$$

and

$$DP : splPMHC \to splPNAMHS.$$

Linearization of pre-namhs's

Using the previous notation we can explain what we mean by "linearization" of a pre-namhs. A *linearized pre-namhs* is a triple $(\mathcal{V}, LGr^W(\mathcal{V}), \varepsilon)$ where $\mathcal{V}$ is a pre-namhs and $LGr^W(\mathcal{V})$ is a split pre-mhc, and

$$\varepsilon : DP(LGr^W(\mathcal{V})) \overset{\cong}{\to} Gr^W(\mathcal{V})$$

is an equivalence in $splPNAMHS$ of split pre-namhs. Concretely this means that for each weight-filtered object in $\mathcal{V}$, we have an linearized associated-graded object. For example, $(LGr^W(V_{DR}), F)$ is a $\mathbf{G}_m$-equivariant filtered hemiperfect complex whose Dold-Puppe is equivalent to $(Gr^W(V_{DR}), F)$ (via ε), and similarly for the real Betti objects.

Define $nLPNAMHS$ to be the $n+1$-category of linearized pre-namhs's; it may be defined as a fiber product of $n+1$-categories in an obvious way, with a diagram that we leave to the reader to draw.

The mixed Hodge complex conditions

We have a notion of *mixed Hodge complex* which (following [14]) is a pre-mhc which satisfies the following conditions:
—the hemiperfect complexes $Tot^{W,F}(\mathcal{C}_{DR})$ over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ and $Tot^W(\mathcal{C}_{B,\mathbf{R}})$ over $\mathcal{A}_{wt,\mathbf{R}}$ are actually perfect complexes;
—strictness of the Hodge filtration on $Gr^W(\mathcal{C})$, a condition which we denote by $\mathbf{Str}(Gr^W(\mathcal{C}), F)$, which actually means that $Tot^F(Gr^W(\mathcal{C}))$ splits (non-canonically) as a product of its cohomology vector bundles over $\mathcal{A}_{hod}$ (cf Lemma 5); and
—the purity condition $\mathbf{MHC}(Gr^W(\mathcal{C}))$ which says that on the k-th graded piece for the $\mathbf{G}_m$-action

$$H^{-i}(Gr_k^W(\mathcal{C})) \subset H^{-i}(Gr^W(\mathcal{C}))$$

the Hodge filtration F and its complex conjugate $\overline{F}$ (with respect to the real structure on the vector space in question), are $k - i$-opposed. In other words, this condition says that the split pre-mhs's $H^{-i}Gr^W(\mathcal{C})$ are $-i$-shifted mhs's. This shifting depending on the cohomological degree, is the fundamental point behind Deligne's definition [14]. See also a recent discussion in Saito [53]. We give a heuristic justification for the sign in the next subsection below.

If one has the strictness condition, then the $H^{-i}Tot^F(Gr^W(\mathcal{C}))$ are vector bundles over $\mathcal{A}_{hod}$, thus

$$(H^{-i}Gr_k^W(\mathcal{C}), F) \in F.VECT$$

are filtered vector spaces in the sense of the discussion of page 16.

We obtain an $n + 1$-category MHC of mixed Hodge complexes.

More generally, we say that a pre-mhc $\mathcal{C}$ is an *s-shifted mhc* if it satisfies the strictness condition and if the $H^{-i}Gr^W(\mathcal{C})$ are $s - i$-shifted mhs's.

The sign in Deligne's shift

Getting the signs right in the condition **MHC** is a nontrivial task. Recall from the introduction that there are three numbers involved: the *Hodge degree* which is the integer h such that the Hodge filtration and its complex conjugate are h-opposed; the *weight* w which is the degree in the weight filtration, equal to k on the graded piece $Gr_k^W(V)$; and the *homotopical degree* i which indicates that we are looking at π_i or H^{-i}. Deligne's shift in the condition **MHC** is the linear relationship

$$h = w - i.$$

We give a heuristic argument here which shows how to determine the sign in this formula.

We know of course that the sign of h should be the same as that of w; the question is whether to put $+$ or $-$ in front of i. In order to resolve this, note that for a coefficient stack V (such as an Eilenberg-MacLane stack $V = K(U, m)$) we have that

$$\pi_0 \underline{Hom}(X_M, V)$$

contains (in a vague way) $H^j(X_M, \pi_j(V))$. Thus we would like to require that

$$h(\pi_0 \underline{Hom}(X_M, V)) = h(H^j(X_M, \pi_j(V)))$$

and
$$w(\pi_0\underline{Hom}(X_M, V)) = w(H^j(X_M, \pi_j(V))).$$

On the other hand, we will be proposing that X_M be constant in the weight-filtration direction, so

$$w(H^j(X_M, \pi_j(V))) = w(\pi_j(V)).$$

Taking the j-th cohomology of X_M has the effect of adding j to the Hodge degree:
$$h(H^j(X_M, \pi_j(V))) = h(\pi_j(V)) + j.$$

Finally, note that $i = 0$ for the π_0, so whatever the sign is, we will have

$$h(\pi_0\underline{Hom}(X_M, V)) = w(\pi_0\underline{Hom}(X_M, V)).$$

Putting all of these together we obtain the formula:

$$h(\pi_j(V)) + j = w(\pi_j(V))$$

or

$$h(\pi_j(V)) = w(\pi_j(V)) - j.$$

As j is the homotopical degree of the piece π_j, this justifies the sign in the formula for condition **MHC**.

Application to linearized pre-namhs's

If $(\mathcal{V}, LGr^W(\mathcal{V}), \varepsilon)$ is an linearized pre-namhs then we say that it satisfies the *strictness condition* $\mathbf{Str}(LGr^W(\mathcal{V}), F)$ if the split pre-mhc $LGr^W(\mathcal{V})$ satisfies the strictness condition (and if the cohomology sheaves are vector bundles, in particular this condition implies that the hemiperfect complexes underlying $LGr^W(\mathcal{V})$ are actually perfect complexes); and we say that it satisfies the *mixed Hodge complex condition* denoted $\mathbf{MHC}(LGr^W(\mathcal{V}))$ if the split pre-mhc $LGr^W(\mathcal{V})$ satisfies the "purity condition" described above.

If these two conditions are satisfied, then the $H^{-i}(LGr^W(\mathcal{V}))$ are split mixed Hodge structures with weight shifted by i. In particular, for $i = 0$ there is no shift and $H^0(LGr^W(\mathcal{V}))$ is a split mixed Hodge structure.

Truncated mixed Hodge complexes

While the Hodge filtration is strict on a mixed Hodge complex (on the Gr^W this is an axiom, and on the whole mhc it is a result of [14] Scholie 8.1.9), the weight filtration is not in general strict. This is seen by the fact that the spectral sequence for W degenerates at E_2 but not necessarily at E_1 (the objects occuring in Proposition 14 p. 68) provide an example of this behavior). A consequence of this is that a shift and truncation of a mixed Hodge complex is no longer a mixed Hodge complex. We will see what type of structure it does have, and this provides motivation for the role of the annihilator ideals in the definition of namhs.

The definition of pre-mhc was made using the notion of hemiperfect complex; recall that this notion is stable under the truncated shift operation Ω; thus the notion of pre-mhc is stable under the truncated shift operation. The conditions **Str** and **MHC** are also stable under the truncated shift operation. In particular, it follows that the associated-graded $(Gr^W(\mathcal{C}), F)$ is a filtered object in $nPERF$. Thus the only filtrations that occur in $nHPERF$ rather than in $nPERF$ are the weight filtrations.

A *truncated mixed Hodge complex* is the same type of thing as a mixed Hodge complex but without the condition that the underlying complexes be perfect (thus they are only hemiperfect cf p. 23), and with the following additional requirements. First, Gr^W is required to be an actual perfect complex and the Hodge filtration here is required to be strict. Then we also require:
A1 that the annihilator ideals are all equal,

$$\forall m \geq 1, \quad Ann^W(t^m; \mathcal{C}) = Ann^W(t; \mathcal{C});$$

and **A3** that the linear subspace defined by this annihilator ideal be a sub-mixed Hodge structure of $H^0(Gr^W(\mathcal{C}))$.

Corollary 11 *Suppose $\mathcal{D}$ is a k-shifted mixed Hodge complex of length $n+k$ and let $\mathcal{C} = \Omega^k \mathcal{D}$. Then $\mathcal{C}$ is a truncated mixed Hodge complex.*

Proof: Recall that the terms

$$H^{-i}(Gr_l^W(\mathcal{C})) = H^{-k-i}(Gr_l^W(\mathcal{D}))$$

are pure Hodge structures with weight $l - i - k + k = l - i$, the first $-i - k$ coming from the shift in the definition of condition **MHC** and the second k coming from the fact that we supposed $\mathcal{D}$ was k-shifted. Thus we get that $\mathcal{C}$ satisfies condition **MHC**. The sheaves entering into condition **Str** are the same as those for $\mathcal{D}$ so $\mathcal{C}$ satisfies condition **Str**.

Recall from [14] that the spectral sequence of the weight filtration on a mixed Hodge complex such as $\mathcal{D}$, degenerates at E_2. Therefore by Lemma 6 the truncated and shifted complex $\mathcal{C}$ satisfies condition **A1**. For condition **A3** the subspace in question is the direct sum over weights l of the kernel of the differential

$$d : H^{-k}(Gr_l^W(\mathcal{D})) \to H^{1-k}(Gr_{l-1}^W(\mathcal{D})).$$

This differential is a morphism of l-shifted pure Hodge structures, so its kernel is again an l-shifted pure Hodge structure. This provides condition **A3** and completes the proof. $|||$

Given a truncated mixed Hodge complex as defined above, we leave as an exercise to show that the operation of taking the flat subobject of the total spaces of the weight filtrations, gives canonically and functorially a mixed Hodge complex. Combining this exercise with the previous corollary, one obtains the result of Hain and Zucker [25] showing how to truncate a mixed Hodge complex into another mixed Hodge complex. Certainly the key conditions **A1** and **A3** must appear in disguised form in the argument of [25].

Nonabelian mixed Hodge structures

A *nonabelian mixed Hodge structure* or *namhs* is a linearized pre-namhs (see page 55) which satisfies a number of conditions which we shall now explain. Before getting to the conditions, we say that the $n + 1$-*category of namhs* denoted $nNAMHS$, is defined as the full sub-$n + 1$-category of $nLPNAMHS$ (see page 55) consisting of those objects which satisfy the following conditions. In particular, a morphism of namhs is defined as a morphism of linearized pre-namhs.

Conditions: These conditions are divided into parts (I), (II) and (III).

$$(I)$$

We impose the conditions that we discussed above, namely:
A1(V_{DR}, W, F) (resp. **A1**$(V_{B,\mathbf{R}}, W)$), that the annihilator ideals
$Ann^W(t^m; Tot^F(V_{DR}))$ (resp. $Ann^W(t^m; V_{B,\mathbf{R}})$) are all equal for $m \geq 1$;
A2(V_{DR}, W, F) (resp. **A2**$(V_{B,\mathbf{R}}, W, F)$), that the annihilator ideal
$Ann^W(t; Tot^F(V_{DR}))$ (resp. $Ann^W(t; V_{B,\mathbf{R}})$) is contained in the ideal of the zero section of $Gr^W(Tot^F(V_{DR}))$ (resp. $Gr^W(V_{B,\mathbf{R}})$);
Fl(V_{DR}, W, F) that the Hodge filtration F is flat.

(II)

These are the mixed Hodge complex conditions. First we ask that:
Str$(LGr^W(V_{DR}), F)$ the sheaves

$$H^{-i}(Tot^F(LGr^W(V_{DR}))/\mathcal{A}_{hod}) \cong \pi_i(Tot^F Gr^W(V_{DR})/\mathcal{A})$$

be vector bundles over $\mathcal{A}_{hod}$. In particular, by Lemma 5, the linearization of $(Tot^F(V_{DR}), W)$ is a perfect one cf p. 26.
Now the object

$$LGr^W(\mathcal{V}) = \{(LGr^W(V_{DR}), F), LGr^W(V_{B,\mathbf{R}}), LGr^W(\zeta_\mathcal{V})\}$$

is basically a graded complex with filtration F and real structure. We can turn this into a complex with a real structure, a real filtration W and a complex filtration F, by setting W equal to the filtration associated to the grading, see p. 18.
MHC$(LGr^W(\mathcal{V}))$ We ask that the resulting object be a mixed Hodge complex [14], see also [53]. Note that this includes the strictness condition **Str** above, plus a "purity" condition. By the strictness condition, the cohomology objects $H^{-i}(LGr^W(\mathcal{V}))$ are vector spaces provided with a Hodge filtration in the stack $VECT$, which just means filtered vector spaces in the usual sense. These vector spaces also have a grading by the $\mathbf{G}_m$-action and a real structure, and the purity condition says that on the k-th graded piece $H^{-i}(LGr_k^W(\mathcal{V}))$ of $H^{-i}(LGr^W(\mathcal{V}))$, the Hodge filtration F and its complex-conjugate $\overline{F}$ (taken with respect to the real structure), are m-opposed with

$$m = k - i.$$

This shift by the homotopical degree i originates with Deligne [14]. In our notation, see the subsection on p. 56 for an explanation of the sign.

In particular the above condition means that the vector space

$$\mathcal{P} := H^0(LGr^W(\mathcal{V}))$$

is a split mixed Hodge structure. To fix notations for below, we have

$$\mathcal{P} = \{(P_{DR} := H^0 LGr^W(V_{DR}), F), P_{B,\mathbf{R}} := H^0 LGr^W(V_{B,\mathbf{R}}), \zeta_P\}$$

together with an action of $\mathbf{G}_m$ on this collection of stuff. Here note that ζ_P is actually an algebraic isomorphism of vector spaces between P_{DR} and

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$P_{B,\mathbf{C}} = P_{B,\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. Thus $\mathcal{P}$ consists of a graded real vector space with a complex filtration F. Turning the grading back into a weight filtration W we obtain a pre-mixed Hodge structure, and one of the consequences of conditions **Str** and **MHC** is that this object $\mathcal{P}$ is a mixed Hodge structure.

<div align="center">(II)</div>

We finally get to our third condition. Recall that $Ann^W(t, \mathcal{V})$ is a sheaf of ideals on $\pi_0(Gr^W(\mathcal{V})) = \mathcal{P}$. The isomorphism ζ_P transforms the de Rham version $Ann^W(t, V_{DR})$ to the complexified Betti version $Ann^W(t, V_{B,\mathbf{C}})$, so we don't need to distinguish between these. Our sheaf of ideals has a Hodge filtration, a real structure, and it is compatible with the $\mathbf{G}_m$-action on $\mathcal{P}$. The $\mathbf{G}_m$-action may be transformed into a weight filtration on $\mathcal{P}$. Thus

$$Ann^W(t; \mathcal{V}) \subset Sym^{\cdot}(\mathcal{P}^*)$$

is a sub-pre-mixed Hodge structure (i.e. subobject with two filtrations and a real structure preserving the weight filtration). The condition of part (III) is stated as our *third condition on the annihilator ideals:*
A3$(\mathcal{V})$, that $Ann^W(t; \mathcal{V}) \subset Sym^{\cdot}(\mathcal{P}^*)$ be a sub-mixed Hodge structure.

This completes our definition of "nonabelian mixed Hodge structure".

Remark: Conditions **Str** and **MHC** are conditions on $LGr^W(V)$, in other words they are conditions only on the "abelian" part; conditions **A1**, **A2**, **A3** and **Fl** concern the nonabelian part.

Homotopy group sheaves

A property which we would clearly like to have is that the homotopy group sheaves of a nonabelian mixed Hodge structure, should carry mixed Hodge structures in a natural way. This turns out to be less obvious than one might think, and requires a somewhat technical treatment. We will give a sketch here. The more technical parts of this section are optional and are not used in the computations in Part III. On the other hand, the first set of remarks—including Proposition 12 and up to the section on Whitehead products p. 64—should be illuminating for understanding what is going on in Part III, so we suggest to read this section lightly the first time.

There is probably a close correspondence between 1-connected nonabelian mixed Hodge structures and the type of "mixed Hodge dga's" considered by Morgan [45] and Hain [26]. In this sense, everything that we say in the present

section should probably be considered as already known by [45] and [26]; and one should consider that our work in this section consists of transcribing those constructions into the context of our present definitions of nonabelian mixed Hodge structure using n-stacks.

Throughout this section, $\mathcal{V}$ will denote a nonabelian mixed Hodge structure which is simply connected, i.e. the n-stacks involved are 1-connected relative to their base stacks $\mathcal{A}$ or $\mathcal{A} \times \mathcal{A}$. In particular, this implies that $\mathcal{V}$ is smooth so the conditions on the annihilator ideals and the flatness assumption on the Hodge filtration are automatic and we don't use them. It is clear that eventually one would like to treat the case of a non-simply connected $\mathcal{V}$ but this is left for a future paper.

Locally over their bases, all of the n-stacks involved in the definition of $\mathcal{V}$ have basepoints, and the basepoints are unique up to locally-defined homotopy which is locally unique up to a 2-homotopy which itself is not unique; all of this is because of our hypothesis that $\mathcal{V}$ is simply connected. In particular there is no need to specify a basepoint section when speaking of the homotopy group sheaves $\pi_i(\mathcal{V})$.

The vector bundle case

The first situation we consider (and this is the one which is encountered in our construction of an example in Part III) is the case when the homotopy group sheaves $\pi_i(\mathcal{V})$ are vector bundles. To explain this condition recall that $\mathcal{V}$ consists essentially of the data of geometric n-stacks

$$Tot^{W,F}(V_{DR}) \to \mathcal{A}_{wt} \times \mathcal{A}_{hod}, \quad Tot^W(V_{B,\mathbf{R}}) \to \mathcal{A}_{wt,\mathbf{R}}$$

together with an analytic equivalence and a linearization. We define $\pi_i(\mathcal{V})$ as being the object consisting of sheaves in the big site

$$Tot^{W,F}(\pi_i(\mathcal{V})_{DR}) := \pi_i(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod}) \quad \text{over} \quad \mathcal{A}_{wt} \times \mathcal{A}_{hod}$$

and

$$Tot^W(\pi_i(\mathcal{V})_{B,\mathbf{R}}) := \pi_i(Tot^W(V_{B,\mathbf{R}})/\mathcal{A}_{wt,\mathbf{R}}) \quad \text{over} \quad \mathcal{A}_{wt,\mathbf{R}}$$

together with an (analytic but which is automatically algebraic) equivalence between $Tot^W(\pi_i(\mathcal{V})_{DR})$ and $Tot^W(\pi_i(\mathcal{V})_{B,\mathbf{C}})$. In general $\pi_i(\mathcal{V})$ is a pre-mixed Hodge object (p. 51) in the stack $VESH$ of vector sheaves, see below. However, our hypothesis in this subsection is that the objects $Tot^{W,F}(\pi_i(\mathcal{V})_{DR})$ and $Tot^W(\pi_i(\mathcal{V})_{B,\mathbf{R}})$ are vector bundles over their respective base objects.

With this hypothesis, $(\pi_i(\mathcal{V})_{DR}, W, F)$ becomes an object of $F.F.VECT$ (i.e. just a bifiltered vector space) and $(\pi_i(\mathcal{V})_{B,\mathbf{R}}, W)$ becomes an object of $F.VECT_{\mathbf{R}}$ (i.e. a filtered real vector space); and the equivalence is just

$$(\pi_i(\mathcal{V})_{DR}, W) \cong (\pi_i(\mathcal{V})_{B,\mathbf{C}}, W).$$

All in all this means that $\pi_i(\mathcal{V})$ is a pre-mixed Hodge structure in the classical sense, i.e. an object of $PMH.VECT$.

A first version of our main result of this section is the following proposition.

Proposition 12 *If $\mathcal{V}$ is a simply connected namhs such that the sheaves underlying the object $\pi_i(\mathcal{V})$ as defined above are vector bundles, then this pre-mixed Hodge structure is a $-i$-shifted mixed Hodge structure.*

Proof: Note that
$$\pi_i Gr^W(\mathcal{V}) = Gr^W(\pi_i(\mathcal{V})),$$

indeed these two objects are obtained by restricting all of the sheaves involved, to $[0] \subset \mathcal{A}_{wt}$. Note also that $Gr^W(\pi_i(\mathcal{V}))$ is the actual associated-graded of the pre-mixed Hodge structure $\pi_i(\mathcal{V})$, in other words it is a vector space with real structure and Hodge filtration obtained by taking the associated-graded of the vector space $\pi_i(\mathcal{V})$ with respect to its weight-filtration by subspaces. The indexation of the components determined by the $\mathbf{G}_m$-action is compatible with the usual one, i.e.

$$\pi_i Gr_k^W(\mathcal{V}) = Gr_k^W(\pi_i(\mathcal{V})) = W_k \pi_i(\mathcal{V})/W_{k-1}\pi_i(\mathcal{V})$$

(see p. 17).

The hypothesis that $\mathcal{V}$ is a namhs means that it is linearized and that the pre-mhc $LGr^W(\mathcal{V})$ a mixed Hodge complex. In particular, this means that the Hodge filtration and its complex conjugate are $k - i$-opposed on

$$H^{-i}(LGr_k^W(\mathcal{V})) = \pi_i(Gr_k^W(\mathcal{V})) = Gr_k^W(\pi_i(\mathcal{V})).$$

But this is exactly the definition of $\pi_i(\mathcal{V})$ being a $-i$-shifted mixed Hodge structure. |||

In the context of these arguments we can briefly say what complicates the general case: in general the $\pi_i(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod})$ are just vector sheaves rather than vector bundles over the base $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$. Thus while they

yield actual filtrations of the underlying vector space $\pi_i(V_{DR})$ by subspaces (cf p. 16), one doesn't recover the total vector sheaves from these filtrations and in particular the fiber of say $\pi_i(Tot^W(V_{DR})/\mathcal{A}_{wt})$ over $[0]$ does not in general correspond to the associated-graded for the filtration by subspaces. The simple argument given for Proposition 12 no longer works.

Normalized homotopy objects

It is convenient to *normalize* the pre-mhs's $\pi_i(\mathcal{V})$ which are $-i$-shifted mhs's, to turn them into actual mixed Hodge structures. To do this, re-index the weight filtration shifting the indexation by i. Call the resulting mixed Hodge structures the *normalized homotopy groups* of $\mathcal{V}$, denoted $\pi_i^{\nu}(\mathcal{V})$.

Whitehead products

Before getting on to the treatment of the general problem raised in the preceding paragraph, we discuss the "Whitehead products" in the easy case represented by Proposition 12.

If (X, x) is a pointed space then we have the functorial *Whitehead product* which is a bilinear map

$$Wh_{i,j} : \pi_i(X, x) \times \pi_j(X, x) \to \pi_{i+j-1}(X, x).$$

The same holds if X is an n-groupoid by the equivalence of [68]. Suppose now that X is an n-stack over a base B (which could be a scheme, a stack or an n-stack itself). Suppose $x : B \to X$ is a basepoint section. Then we obtain a bilinear map of sheaves over B

$$Wh_{i,j} : \pi_i(X/B, x) \times \pi_j(X/B, x) \to \pi_{i+j-1}(X/B, x).$$

If X is relatively 1-connected over B then locally over B there is a basepoint section and the homotopy groups don't depend on choice of basepoint section up to unique isomorphism, so we obtain relative $\pi_i(X/B)$ which are sheaves over B (which if necessary means morphisms from B to the 1-stack SH of sheaves). These exist even though there need not exist a global section $x : B \to X$. In this case the Whitehead product maps again give bilinear maps of sheaves over B,

$$Wh_{i,j} : \pi_i(X/B) \times \pi_j(X/B) \to \pi_{i+j-1}(X/B).$$

Now we return to the case of a 1-connected namhs $\mathcal{V}$. We obtain bilinear maps of the sheaves occuring in the definitions of the $\pi_i(\mathcal{V})$, for example

$$Wh_{i,j} : \pi_i(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod}) \times \pi_j(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod})$$

$$\rightarrow \pi_{i+j-1}(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod}).$$

Idem for $V_{B,\mathbf{R}}$, and these maps are compatible with the equivalences ζ.

In the case we are currently considering, the terms in the above bilinear map are assumed to be vector bundles over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$. Thus the bilinear map becomes a morphism from the tensor product of the two bundles, to the third one. The correspondence between filtered vector spaces in the usual sense and bundles over $\mathcal{A}$ is compatible with tensor product. Taking the tensor product of all of the filtrations in question, we obtain a notion of tensor product of pre-mhs. The Whitehead product becomes a morphism of pre-mhs

$$Wh_{i,j}^{\mathrm{pre}}(\mathcal{V}) : \pi_i(\mathcal{V}) \otimes \pi_j(\mathcal{V}) \rightarrow \pi_{i+j-1}(\mathcal{V}).$$

The alert reader will have noticed that there is a problem with the shifts here: on the left we have tensored a $-i$-shifted mhs with a $-j$-shifted mhs, which gives a $-i-j$-shifted mhs. On the right is a $1-i-j$-shifted mhs.

Here is where our main idea, of requiring that the associated-graded be a linear object, comes in. The object $Gr^W(\mathcal{V})$ is linear, so its Whitehead products vanish. In other words, the associated-graded of the above Whitehead product for $\mathcal{V}$ vanishes:

$$Gr^W(Wh_{i,j}^{\mathrm{pre}}(\mathcal{V})) = 0.$$

In other words, $Wh_{i,j}^{\mathrm{pre}}(\mathcal{V})$ lowers the degree in the weight filtration by one. Denote by $\pi_{i+j-1}(\mathcal{V})(-D_{wt})$ the pre-mhs obtained by shifting the weight filtration by one, in the direction such that there is a morphism of pre-mhs inducing zero on the associated-graded,

$$\pi_{i+j-1}(\mathcal{V})(-D_{wt}) \rightarrow \pi_{i+j-1}(\mathcal{V}).$$

Then $\pi_{i+j-1}(\mathcal{V})(-D_{wt})$ is a $-i-j$-shifted mhs, and the Whitehead product factors through a morphism

$$Wh_{i,j}(\mathcal{V}) : \pi_i(\mathcal{V}) \otimes \pi_j(\mathcal{V}) \rightarrow \pi_{i+j-1}(\mathcal{V})(-D_{wt}).$$

This is now a morphism of $-i-j$-shifted mixed Hodge structures.

If we normalize the homotopy group sheaves, and include an extra shift by one to normalize $\pi_{i+j-1}(\mathcal{V})(-D_{wt})$ (which gives the same answer as if we normalize $\pi_{i+j-1}(\mathcal{V})$), then we have ended up shifting by $i+j$ on both sides of the map $Wh_{i,j}(\mathcal{V})$, so it normalizes to a morphism of mixed Hodge structures (the *Whitehead product*):

$$Wh_{i,j}^{\nu}(\mathcal{V}) : \pi_i^{\nu}(\mathcal{V}) \otimes \pi_j^{\nu}(\mathcal{V}) \to \pi_{i+j-1}^{\nu}(\mathcal{V}).$$

The fact that the Whitehead product goes from degree i times degree j to degree $i+j-1$ means that in order to get a correctly shifted morphism of mixed Hodge structures, we need to pick up a shift by 1 somewhere. This shift by 1 comes from the fact that by assumption the Whitehead products vanish on $Gr^W(\mathcal{V}) = DP(LGr^W(\mathcal{V}))$. This situation may inversely be taken as one of the motivations for the idea of requiring $Gr^W(\mathcal{V})$ to be a linear object.

The general case

We now turn to the general case where we don't make the hypothesis that the sheaves underlying $\pi_i(\mathcal{V})$ are vector bundles. Recall from [62] that if $X \to B$ is a relatively 1-connected geometric n-stack over a base B then the $\pi_i(X/B)$ are vector sheaves over B (which if necessary means morphisms $B \to VESH$ to the 1-stack of vector sheaves). Recall also from pages 16 and 17 that if U is a vector sheaf over $\mathcal{A}$ then we obtain a filtration $W_{\cdot}$ of the vector space $U_{[1]}$ by subspaces defined by the condition that $u \in W_k$ if and only if $t^{-k}\tilde{u}$ extends to a section of U over $\mathbf{A}^1$. This functor from vector sheaves over $\mathcal{A}$ to filtered vector spaces is unfortunately not an equivalence of categories, and the fiber $U_{[0]}$ is no longer identifiable with the usual associated-graded of the filtration (i.e. the direct sum of the subspaces W_k/W_{k-1}). This is what makes the present situation more complicated.

We continue to assume that $\mathcal{V}$ is a nonabelian mixed Hodge structure which is 1-connected. Thus the sheaves underlying $\pi_i(\mathcal{V})$ are vector sheaves over the appropriate base stacks $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ or $\mathcal{A}_{wt,\mathbf{R}}$. We obtain in particular, vector sheaves

$$Tot^F(\pi_i(V_{DR})) \quad \text{over} \quad \mathcal{A}_{hod},$$

and

$$Tot^W(\pi_i(V_{B,\mathbf{R}})) \quad \text{over} \quad \mathcal{A}_{hwt,\mathbf{R}}.$$

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The fibers of these over [1] are respectively the finite dimensional complex vector space $\pi_i(V_{DR})$ and the finite dimensional real vector space $\pi_i(V_{B,\mathbf{R}})$; and the former is isomorphic via ζ to the complexification of the latter. Let $\pi_i(V)$ denote these isomorphic spaces, together with its real structure. From the above vector sheaves we obtain filtrations $F^{\cdot}$ (complex) and $W_{\cdot}$ (real) of $\pi_i(V)$ by subspaces (again see pages 16 and 17).

This yields a pre-mixed Hodge structure

$$\pi_i^\mu(\mathcal{V}) := \{\pi_i(V), \pi_i(V)_{\mathbf{R}}, F^{\cdot}, W_{\cdot}\}.$$

Our goal is to obtain the following result.

Theorem 2 *The pre-mixed Hodge structure $\pi_i^\mu(\mathcal{V})$ which we have defined above is a $-i$-shifted mhs. Denote by $\pi_i^\nu(\mathcal{V})$ its version where the weight filtration is shifted by i places so that it becomes a mixed Hodge structure. The Whitehead products on the homotopy groups of V_{DR} give morphisms of mixed Hodge structures*

$$Wh_{i,j}^\nu : \pi_i^\nu(\mathcal{V}) \otimes \pi_j^\nu(\mathcal{V}) \to \pi_{i+j-1}^\nu(\mathcal{V}).$$

Along the way we will also obtain a somewhat more precise structure theorem about the vector sheaves underlying the original object $\pi_i(\mathcal{V})$.

The case of mixed Hodge complexes

Suppose $\mathcal{C}$ is a mixed Hodge complex. Its Dold-Puppe $\mathcal{V} = DP(\mathcal{C})$ is a namhs, and the homotopy groups of the latter are the same as the cohomology groups of $\mathcal{C}$. Taking the cohomology sheaves of all of the perfect complexes underlying $\mathcal{C}$ we obtain a "pre-mixed Hodge vector sheaf" $H^{-i}(\mathcal{C})$ and the Dold-Puppe compatibility says that this object is equivalent to the object $\pi_i(\mathcal{V})$ considered above. In particular, we encounter the same problem as that which was outlined above, for the $H^{-i}(\mathcal{C})$. On the other hand, Deligne shows in [14] that the cohomology groups of a mixed Hodge complex are (shifted) mixed Hodge structures. Our strategy for proving Theorem 2 will be to reduce to the case of a mixed Hodge complex. For this, we need a structure theorem which says that in a certain sense one can "split off" the behavior at the rightmost part of the mixed Hodge complex (which corresponds to the lowest homotopy degrees). From there, we will be able to use the Hurewicz fact that homotopy and homology coincide in low degrees to apply the case of mixed Hodge complexes to the general case.

The following two propositions and the resulting theorem represent our "splitting-off" results. As they are results about mixed Hodge complexes, we leave their proofs as exercises for the reader, to be done using what is known e.g. in Deligne [14], Hain and Zucker [25], Saito [53] [54] [55] etc.

Proposition 13 *Suppose $\mathcal{C}$ is a mixed Hodge complex which is supported in $(-\infty, -k]$. Let C denote the underlying complex (i.e. one of the two isomorphic complexes C_{DR} or $C_{B,\mathbf{C}}$). If $H^{-k}(C) \neq 0$ then $H^{-k}(C)$ provided with its Hodge and weight filtrations and real structure, is a $-k$-shifted mixed Hodge structure. Furthermore, if we let $H_\mu^{-k}(\mathcal{C})$ denote this shifted mixed Hodge structure considered as a mixed Hodge complex sitting in degree $-k$, then there is a morphism of mixed Hodge complexes inducing the identity on $H^{-k}(C)$:*

$$\mathcal{C} \to H_\mu^0(\mathcal{C}).$$

‖‖

Proposition 14 *Suppose $\mathcal{C}$ is a mixed Hodge complex which is supported in $(-\infty, -k]$. Let C denote the underlying complex (i.e. one of the two isomorphic complexes C_{DR} or $C_{B,\mathbf{C}}$). Suppose that $H^{-k}(C) = 0$. Then there is a split $-k$-shifted mixed Hodge structure $K = H^{-k}(Gr^W(\mathcal{C}))$, and a mixed Hodge complex $\mathcal{K}$ supported in $[-k-1, -k]$ such that $H^{-k}(Gr^W(\mathcal{K})) = K$, such that $H^{-1-k}(Gr^W(\mathcal{K})) = K'$ is the split $-1-k$-shifted mixed Hodge structure obtained by shifting the weight filtration of K by one, and such that the complex underlying $\mathcal{K}$ is exact. The differential of $\mathcal{K}$ is an isomorphism of pre-mixed Hodge complexes but which shifts the weight filtration by one. There is a morphism of mixed Hodge complexes*

$$f : \mathcal{C} \to \mathcal{K}$$

which induces an isomorphism $H^{-k}(Gr^W(\mathcal{C})) \cong H^{-k}(Gr^W(\mathcal{K}))$. Furthermore, $Cone(f)$ is quasi-isomorphic to a mixed Hodge complex supported in $[-\infty, -1-k]$.

‖‖

We show how to use these two propositions to understand the structure of $H^i(\mathcal{C})$. Suppose $\mathcal{C}$ is supported in $(-\infty, -k]$. If $H^{-k}(C) \neq 0$ then we apply

Proposition 13 to obtain a mixed Hodge complex $\mathcal{L}$ supported in degree $-k$ and a morphism

$$g : \mathcal{C} \to \mathcal{L}.$$

Let $\mathcal{C}_1 := Cone(g)$, and let C_1 denote its underlying complex. (If $H^{-k}(C) = 0$ then set $\mathcal{C}_1 = \mathcal{C}$ and pass directly to the next step.) Think of the cone as a kernel of a map, so it is supported in $(-\infty, -k]$. By the property obtained in Proposition 13, $H^{-k}(C_1) = 0$. Therefore we can apply Proposition 14 to obtain a mixed Hodge complex $\mathcal{K}$ supported in $[-1-k, -k]$ and a morphism

$$f : \mathcal{C}_1 \to \mathcal{K},$$

such that $\mathcal{C}_2 := Cone(f)$ is quasiisomorphic to a mixed Hodge complex supported in $(-\infty, -1-k]$. Thus in two steps we have split off two types of mixed Hodge complexes: the first one equal to a mixed Hodge structure supported in a single degree; and the second one supported in two adjacent degrees and whose underlying complex is exact. After doing this, we move the rightmost bound for the interval of support, back by one.

We obtain the following structure theorem for the objects $H^{-i}(\mathcal{C})$.

Theorem 3 *Suppose $\mathcal{C}$ is a mixed Hodge complex supported in $(-\infty, 0]$. Let $H^{-i}(\mathcal{C})$ denote cohomology of $\mathcal{C}$ considered as a pre-mixed Hodge object in the category of vector sheaves. Let $H_\mu^{-i}(\mathcal{C})$ denote the bifiltered vector space with real structure (i.e. pre-mhs) obtained by applying the construction of filtrations of p. 16 as above. Then there are objects $\mathcal{K}$ and $\mathcal{K}'$ (pre-mixed Hodge objects in the category of vector sheaves) and morphisms*

$$a : H^{-i}(\mathcal{C}) \to \mathcal{K} \to 0,$$

$$0 \to \mathcal{K}' \xrightarrow{b} H^{-i}(\mathcal{C})$$

*such that $\ker(a)/\operatorname{im}(b) \cong H_\mu^{-i}(\mathcal{C})$. Furthermore, the object $Tot^{W,F}(\mathcal{K})$ is a vector scheme over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$, satisfying property **A1**, whose fiber over $[1] \times \mathcal{A}_{hod}$ is 0 and whose fiber over $[0] \times \mathcal{A}_{hod}$ corresponds to a split $-i$-shifted mixed Hodge structure. Dually, the object $Tot^{W,F}(\mathcal{K}')$ is a coherent sheaf over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ whose dual vector scheme satisfies property **A1** (this means that the coherent sheaf is annihilated by the coordinate t on $\mathcal{A}_{wt}$). Again its fiber over $[1] \times \mathcal{A}_{hod}$ is 0 and its fiber over $[0] \times \mathcal{A}_{hod}$ corresponds to a split $-i$-shifted mixed Hodge structure. Finally, the differential of the mixed Hodge complex is an isomorphism between the split shifted mhs corresponding to $\mathcal{K}$ for degree $-1-i$, and the split shifted mhs corresponding to $\mathcal{K}'$ for degree i, except that this isomorphism shifts the weight filtration by one.*

Homology and cohomology of a geometric n-stack

Suppose X is a 1-connected geometric n-stack relative to a base B. Then we obtain its *cohomology object* $\mathbf{H}^*(X/B, \mathcal{O})$ which is a perfect complex over B, supported in $[0, \infty)$. This may be seen as the higher direct image complex $\mathbf{R}p_*(\mathcal{O})$ where $p : X \to B$ is the projection.

Denote the dual perfect complex by

$$\mathbf{H}_*(X/B) := \underline{Hom}(\mathbf{H}^*(X/B, \mathcal{O}), \mathcal{O}).$$

It is a perfect complex over B, supported in $(-\infty, 0]$. Its Dold-Puppe is an ∞-stack over B such that any truncation is "almost geometric" in the sense of [66] p. 118. We have a morphism

$$s : X \to DP(\mathbf{H}_*(X/B)).$$

This morphism is the "stabilization map", and one could write

$$DP(\mathbf{H}_*(X/B)) = \Omega^\infty S^\infty(X/B).$$

(While we don't go into that here, one could make a precise statment of a lemma represented by the previous equation.)

The version of Hurewicz which applies to the present situation is the following lemma:

Lemma 15 *Suppose X is $k-1$-connected relative to B, for $k \geq 2$. Then the above stabilization map s induces an isomorphism between the homotopy vector sheaf $\pi_k(X/B)$ and the cohomology vector sheaf $H^{-k}(\mathbf{H}_*(X/B))$. In particular, the homotopy fiber of s is k-connected.*

Homology of a namhs

We now turn back to consideration of a 1-connected nonabelian mixed Hodge structure $\mathcal{V}$. We apply the discussion of the previous subsection to the geometric n-stacks underlying $\mathcal{V}$, for example $Tot^{W,F}(V_{DR}) \to \mathcal{A}_{wt} \times \mathcal{A}_{hod}$.

We obtain homology objects such as

$$\mathbf{H}_*(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod})$$

which is a perfect complex over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ supported in $(-\infty, -2]$. Doing this for $Tot^W(V_{B,\mathbf{R}})$ and the equivalence ζ too, we obtain all in all a pre-mixed Hodge object in the ∞-stack of perfect complexes supported in $(-\infty, -2]$. Denote this object by $\mathbf{H}_*(\mathcal{V})$. It is a pre-mixed Hodge complex supported in $(-\infty, -2]$.

Lemma 16 *Suppose $\mathcal{V}$ is a 1-connected namhs. Then the pre-mixed Hodge complex $\mathbf{H}_*(\mathcal{V})$ constructed above is a mixed Hodge complex.*

Proof: The associated-graded of $\mathbf{H}_*(\mathcal{V})$ is the homology object of $Gr^W(\mathcal{V})$. But since this latter is by hypothesis linearized, and since the Hodge filtration satisfies the strictness hypothesis **Str**, we obtain a noncanonical decomposition of $Gr^W(\mathcal{V})$ into a product of Eilenberg-MacLane objects for split shifted mixed Hodge structures. The homology object (which is a coalgebra) is the tensor product of the homology objects for the Eilenberg-MacLane factors. Each factor looks like $K(\mathcal{U}, i)$ where $\mathcal{U}$ is a split $-i$-shifted mixed Hodge structure. We have the formulae

$$\mathbf{H}_*(K(\mathcal{U}, i)) = Sym^*(\mathcal{U}[i]) \quad \text{for } i \text{ even,}$$

$$\mathbf{H}_*(K(\mathcal{U}, i)) = \overset{*}{\bigwedge}(\mathcal{U}[i]) \quad \text{for } i \text{ odd,}$$

where $\mathcal{U}[i]$ is the object $\mathcal{U}$ placed in degree $-i$. For these formulae, see the "Breen calculations" in Part III below, page 107. From these, it is immediate that $\mathbf{H}_*(Gr^W(\mathcal{V}))$ is a split mixed Hodge complex. The mixed Hodge complex conditions concern only Gr^W so this implies that $\mathbf{H}_*(\mathcal{V})$ is a mixed Hodge complex. |||

The i-th cohomology of the mixed Hodge complex of cohomology $\mathbf{H}^*(\mathcal{V})$ is a pre-mixed Hodge object in $VESH$, and as usual we can extract from it a pre-mhs by taking the induced filtrations on the underlying vector spaces. This yields in fact an i-shifted mixed Hodge structure which we can then normalize by shifting the weight filtration to give a mixed Hodge structure denoted $H^i_\nu(\mathcal{V})$. Cup product is a morphism of mixed Hodge structures, and we expect a statement about compatibility between the higher-order Massey products and this mixed Hodge structure, but we don't have a precise statement at the present.

Homotopy fibers of certain morphisms of namhs

In general, the condition of being a namhs is not stable under homotopy fiber products. To see this one just has to look at the case of a mixed Hodge complex equal to a single $-i$-shifted mhs placed in degree $-i$. Taking the loop-space of the associated namhs (which is a homotopy fiber product of two times the punctual namhs, over the namhs in question) yields the same $-i$-shifted mhs but now placed in degree $1-i$. It is no longer a mixed Hodge complex.

Nonetheless, we need to have a result about preservation of the namhs condition when we take fibers of certain morphisms. A quick look at the long exact sequence for homotopy groups (on the level of Gr^W, say) leads to the appropriate condition: the connecting morphisms in the long exact sequence are morphisms between shifted mhs's with different shifts—there will be a problem if these connecting morphisms are nonzero, so we make a hypothesis that guarantees that the connecting morphisms are zero. We obtain the following statement.

Lemma 17 *Suppose* $f : \mathcal{V} \to \mathcal{U}$ *is a morphism of* 1*-connected nonabelian mixed Hodge structures. Suppose that* $\mathcal{U}$ *is pointed by a point* u. *Suppose that for all* i *the morphism induced by* $Gr^W(f)$ *is a surjection of homotopy group objects:*
$$Gr^W(f) : \pi_i Gr^W(\mathcal{V}) \to \pi_i Gr^W(\mathcal{U}) \to 0.$$
Then the homotopy fiber $\mathcal{F} := \mathcal{V} \times_{\mathcal{U}} u$ *of* f *over* u *is a nonabelian mixed Hodge structure.*

Proof: The n-stacks underlying $\mathcal{F}$ are geometric and relatively 1-connected; therefore they are smooth over the respective base schemes so the annihilator ideals vanish and we have automatically conditions **A1**, **A2**, **A3**. Also locally any basepoint section provides a smooth surjection from a scheme so the flatness condition **Fl** holds too. Thus we just have to verify that $LGr^W(\mathcal{F})$ is a mixed Hodge complex, which contains conditions **Str** and **MHC**. The long exact sequence for the fiber of a morphism yields a long exact sequence (of split pre-mixed Hodge objects in the stack of vector sheaves)

$$\ldots \to H^{-i}(LGr^W(\mathcal{F})) \to H^{-i}(LGr^W(\mathcal{V})) \to H^{-i}(LGr^W(\mathcal{U})) \to \ldots.$$

The morphisms going from the places occupied by $\mathcal{V}$ to those occupied by $\mathcal{U}$, are surjections by hypothesis. Thus the connecting morphisms (the leftmost

and rightmost arrows in the above diagram) are zero and

$$H^{-i}(LGr^W(\mathcal{F})) = \ker\left(H^{-i}(LGr^W(\mathcal{V})) \to H^{-i}(LGr^W(\mathcal{U}))\right).$$

The total objects of the Hodge filtrations

$$Tot^F(H^{-i}(LGr^W(\mathcal{V})) \quad \text{and} \quad Tot^F(H^{-i}(LGr^W(\mathcal{U})))$$

are vector bundles over $\mathcal{A}_{hod}$, therefore the kernel is a vector bundle too. This gives condition **Str** for $\mathcal{F}$. Furthermore the split pre-mhs corresponding to $H^{-i}(LGr^W(\mathcal{F}))$ is the kernel of a morphism of $-i$-shifted mhs's, so it is again a $-i$-shifted mhs. This gives condition **MHC**. We obtain that $\mathcal{F}$ is a namhs. |||

Conclusion: the proof of Theorem 2

We now put together all of the above statements to obtain a proof of Theorem 2. Along the way we will see that the structure result of Theorem 3 also holds for the homotopy group objects $\pi_i(\mathcal{V})$.

Suppose that $\mathcal{V}$ is $k-1$-connected for $k \geq 2$. The proof will be by descending induction on k, so we assume that Theorem 2 and indeed the structure result analogous to Theorem 3 are known for any k-connected namhs $\mathcal{F}$. (To be technically correct in the induction, we add to the inductive hypothesis that at the bottom degree $k+1$ there is no quotient object in the structure result.)

Let $\mathcal{C} := \mathbf{H}_*(\mathcal{V})$. It is a mixed Hodge complex supported in $(-\infty, -k)$. Combining the constructions of Propositions 13 and 14, we obtain a mixed Hodge complex $\mathcal{N}$ supported in $[-1-k, -k]$ and a morphism $\mathcal{C} \to \mathcal{N}$ inducing surjections on the cohomology objects of $Gr^W(\mathcal{C})$ and $Gr^W(\mathcal{N})$, and indeed inducing an isomorphism of cohomology objects in degree $-k$.

The complex underlying $\mathcal{N}$ is exact in degree $-1-k$ and its cohomology in degree $-k$ carries a mixed Hodge structure $H_\mu^{-k}(\mathcal{C})$. The cohomology objects $H^{-i}(\mathcal{N})$ are as follows: in degree $-1-k$, an object such as was denoted $\mathcal{K}$ in Theorem 3; and in degree $-k$ an extension with subobject as was denoted $\mathcal{K}'$ in Theorem 3 and quotient object $H_\mu^{-k}(\mathcal{C})$.

The Dold-Puppe $DP(\mathcal{N})$ has a structure of namhs (it is linearized by setting $LGr^W(DP(\mathcal{N})) := Gr^W(\mathcal{N})$). Note that, whereas $DP(\mathcal{C})$ is an ∞-stack, $DP(\mathcal{N})$ is an n-stack (or at most an $n+1$-stack if we are dealing with the limit case $k = n$).

Now the fact that $k \geq 2$ means that the nonlinear terms in the Breen calculations for the homology of $Gr^W(\mathcal{V})$ start intervening only in degree $k + 2$ or more. In particular the morphism

$$Gr^W(\mathcal{V}) \to DP(Gr^W(\mathcal{C}))$$

induces a surjection on π_k and π_{k+1}. Therefore the morphism $f : \mathcal{V} \to DP(\mathcal{N})$ induces surjections on the homotopy group objects of the associated-graded, in particular f satisfies the hypotheses of Lemma 17. That lemma says that the fiber $\mathcal{F}$ of f over the zero-section of $DP(\mathcal{N})$ is a nonabelian mixed Hodge structure. The long exact sequence of homotopy objects gives a long exact sequence of pre-mixed Hodge objects in the category of vector sheaves,

$$\ldots \to \pi_i(\mathcal{F}) \to \pi_i(\mathcal{V}) \to H^{-i}(\mathcal{N}) \to \pi_{i-1}(\mathcal{F}) \to \ldots.$$

Our inductive hypothesis says that for $i \geq k+2$ the objects $\pi_i(\mathcal{F})$ satisfy the structure result of Theorem 3 (and in particular that the pieces corresponding to bifiltered vector spaces with real structure, are i-shifted mhs's); and it says that for $i = k + 1$ the same holds but with the quotient being zero in the structure result.

Note that for $i \geq k + 2$, $H^{-i}(\mathcal{N}) = 0$ so $\pi_i(\mathcal{V}) = \pi_i(\mathcal{F})$ and we obtain the desired structure result for $\pi_i(\mathcal{V})$. For $i = k + 1$ we have an exact sequence

$$0 \to \pi_{k+1}(\mathcal{F}) \to \pi_{k+1}(\mathcal{V}) \to H^{-1-k}(\mathcal{N}) \to 0.$$

The fact that the connecting map is zero comes from the fact that we constructed $\mathcal{N}$ so that $\pi_{k+1}(\mathcal{V}) \to H^{-1-k}(\mathcal{N})$ is surjective. From this exact sequence we obtain the desired structure result for $\pi_{k+1}(\mathcal{V})$: the inductive hypothesis gives the structure result but without quotient object, for $\pi_{k+1}(\mathcal{F})$; and the piece $H^{-1-k}(\mathcal{N})$ provides exactly a quotient object as forseen in Theorem 3. Finally we have by construction an isomorphism

$$\pi_k(\mathcal{V}) \cong H^{-k}(\mathcal{N})$$

(which justifies why $\mathcal{F}$ was k-connected). This gives the structure result for $\pi_k(\mathcal{V})$ with no quotient object, and in particular with $\pi_k^\mu(\mathcal{V})$ being a $-k$-shifted mixed Hodge structure.

By induction we obtain the structure result of Theorem 3 for the homotopy group objects $\pi_i(\mathcal{V})$. This gives the first part of the statement of

Theorem 2, that the $\pi_i^\mu(\mathcal{V})$ are i-shifted mixed Hodge structures. It also gives the statement that the associated-graded objects $Gr^W \pi_i^\mu(\mathcal{V})$ are sub-quotients of the $\pi_i(Gr^W(\mathcal{V}))$. Now by the existence of a linearization, the Whitehead products between the $\pi_i(Gr^W(\mathcal{V}))$ vanish. This implies that the Whitehead products between the $Gr^W \pi_i^\mu(\mathcal{V})$ vanish. The same argument as above (starting page 64) gives that on the normalized mixed Hodge structures $\pi_i^\nu(\mathcal{V})$, the Whitehead products are morphisms of mixed Hodge structures. This completes the proof of Theorem 2. |||

We conclude this section by remarking that the above Lemma 17 and its counterexample show that the the $n+1$-category $nNAMHS$ is not closed under homotopy limits, but only under very special kinds of them. We wonder about the following:

Question: Is the $n+1$-category $nNAMHS$ closed under a reasonable class of homotopy *colimits*?

The basic construction

The basic construction which we would like to describe is the "nonabelian cohomology" of a smooth projective variety X with coefficients in a non-abelian mixed Hodge structure. In this section we define the cohomology with coefficients in a pre-namhs $\mathcal{V}$ (eventually linearized), denoted

$$\mathcal{H} = \underline{Hom}(X_M, \mathcal{V}).$$

This will be an object of the same type as a pre-namhs, i.e. an object in $PMH.nSTACK$. We show that it is in fact a pre-namhs (the meaning of this condition being that the n-stacks involved are geometric or equivalently that it is an object of $PMH.nGEOM$) under certian reasonable hypotheses on $\mathcal{V}$. If $\mathcal{V}$ was linearized then $\mathcal{H}$ will be linearized.

In what follows X will be a smooth projective variety over $\mathbf{C}$ and $\mathcal{V}$ will be a pre-namhs (with the same notation as in the definition page 50). As a matter of notation, we will denote by $\mathcal{H}$ the object $\underline{Hom}(X_M, \mathcal{V})$ which we will construct, and as above we will have

$$\mathcal{H} = \{(H_{DR}, W, F), \ (H_{B,\mathbf{R}}, W), \ \zeta_\mathcal{H}\}.$$

We first describe what is meant by X_M. This is an object which is somewhat like a pre-namhs, however the n-stacks involved are not geometric, and the morphism ζ is not an equivalence.

Recall from [60] [66] that we have defined a formal category X_{Hod} over $\mathcal{A}_{hod}$ which gives a 1-stack

$$X_{Hod} \to \mathcal{A}_{hod}.$$

Technically speaking in [60] we denoted by X_{Hod} the $\mathbf{G}_m$-equivariant formal category over $\mathbf{A}^1_{hod}$; for our purposes here it is more convenient to take the quotient and denote by X_{Hod} the object over $\mathcal{A}_{hod}$.

The fiber of X_{Hod} over $[1]$ is X_{DR} and the fiber over $[0] = B\mathbf{G}_m$ is X_{Dol} with its action of $\mathbf{G}_m$. On the other hand, we define $X_{B,\mathbf{R}}$ to be the constant n-stack over $\mathcal{Z}_{\mathbf{R}}$ whose value is the Poincaré n-groupoid corresponding to the n-truncation of the homotopy type of X^{top}. Note that the complexified stack associated to a constant real stack is again a constant stack on $\mathcal{Z}$ with the same value, in particular $X_{B,\mathbf{C}}$ is the constant stack on $\mathcal{Z}$ whose value is the Poincaré n-groupoid of X^{top}.

In order to define the morphism of analytic n-stacks

$$\zeta_X : X^{\text{an}}_{DR} \to X^{\text{an}}_{B,\mathbf{C}},$$

we need the following lemma.

Lemma 18 *Define the n-prestack $C^{\text{pre}}(-, X^{\text{top}})$ on $\mathcal{Z}^{\text{an}}$ to be the prestack which to each Y associates the Poincaré n-groupoid of the space of continuous maps from Y^{top} to X^{top}. Then the associated n-stack which we denote by $C(-, X^{\text{top}})$, is naturally equivalent to $X^{\text{an}}_{B,\mathbf{C}}$.*

Proof: Let $X_{B,pre,\mathbf{C}}$ be the constant n-prestack with values equal to the Poincaré n-groupoid of X^{top}, and let $X^{\text{an}}_{B,pre,\mathbf{C}}$ be the associated analytic prestack which is again a constant prestack. Then $X_{B,\mathbf{C}}$ (resp. $X^{\text{an}}_{B,\mathbf{C}}$) is the n-stack associated to $X_{B,pre,\mathbf{C}}$ (resp. $X^{\text{an}}_{B,pre,\mathbf{C}}$). We have a morphism of prestacks obtained by considering a point of X^{top} as a constant map from any Y^{top} to X^{top},

$$X^{\text{an}}_{B,pre,\mathbf{C}} \to C^{\text{pre}}(-, X^{\text{top}}).$$

This morphism is an equivalence, over any Y such that Y^{top} is contractible. Since the contractible open sets form a base for the topology of any $Y \in \mathcal{Z}^{\text{an}}$, this property implies that the above morphism of prestacks induces an equivalence on associated n-stacks

$$X^{\text{an}}_{B,\mathbf{C}} \xrightarrow{\cong} C(-, X^{\text{top}}).$$

Recall that

$$X_{DR}^{\mathrm{an}}(Y) = X(Y^{\mathrm{red}})$$

where Y^{red} is the reduced complex analytic space. A morphism from Y^{red} to X gives a continuous map from Y^{top} to X^{top}, so we get a morphism

$$X_{DR}^{\mathrm{an}}(Y) \to C(-, X^{\mathrm{top}})(Y) \cong X_{B,\mathbf{C}}^{\mathrm{an}}(Y).$$

Our ζ_X is defined to be the composed morphism $X_{DR}^{\mathrm{an}} \to X_{B,\mathbf{C}}^{\mathrm{an}}$.

The object which we denote informally by X_M is the collection

$$X_M = \{X_{Hod} \to \mathcal{A}_{hod}, X_{B,\mathbf{R}}, \zeta_X\}.$$

Note that X_{Hod} is an object of $F.nSTACK$. We can consider it as an object of $F.F.nSTACK$ by making it constant in the other "weight" direction; and similarly the n-stack $X_{B,\mathbf{R}}$ may be thought of as an object of $F.nSTACK$ constant over $\mathcal{A}_{wt}$. In this sense the object X_M is somewhat similar to a pre-namhs. The important differences are:

–the n-stacks involved are not geometric; and

–the morphism ζ_X is a morphism but not an equivalence.

In spite of these differences, we still define an object $\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$ as we shall now start to do.

The nonlinearized case

We will first construct $\mathcal{H}$ in the case where the pre-namhs $\mathcal{V}$ is not linearized. Put:

$$H_{DR} := \underline{Hom}(X_{DR}, V_{DR}).$$

It is a geometric n-stack by [59], [62].

As indicated above we obtain objects which are constant in the weight direction

$$X_{Hod} \times \mathcal{A}_{wt} \to \mathcal{A}_{hod} \times \mathcal{A}_{wt},$$

and

$$X_{B,\mathbf{R}} \times \mathcal{A}_{wt,\mathbf{R}} \to \mathcal{A}_{wt,\mathbf{R}}.$$

Restricting the first to $[1] \in \mathcal{A}_{hod}$ we obtain

$$X_{DR} \times \mathcal{A}_{wt} \to \mathcal{A}_{wt}.$$

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Now we define the weight and Hodge filtrations on this n-stack by defining the total space as

$$Tot^{W,F}(H_{DR}) := \underline{Hom}^{\mathrm{se},0}\left(\frac{X_{Hod} \times \mathcal{A}_{wt}}{\mathcal{A}_{hod} \times \mathcal{A}_{wt}}, \frac{Tot^{W,F}(V_{DR})}{\mathcal{A}_{hod} \times \mathcal{A}_{wt}}\right).$$

Here $\underline{Hom}(\frac{A}{B}, \frac{C}{B})$ is the relative internal Hom of n-stacks over B. It can also be viewed as the internal Hom in the stack $(n+1)STACK_{1/}(B, nSTACK)$ (there is a compatibility between these two things, which we don't get into).

The member on the left of the internal $\underline{Hom}$ is a formal category of smooth type over the base $B = \mathcal{A}_{hod} \times \mathcal{A}_{wt}$.

The superscript means that we take the substack of *semistable morphisms with vanishing Chern classes*, see the discussion in [66] §10.7 and on page 85 below. This semistability condition is automatic for Betti or de Rham cohomology, so it is only necessary for the Dolbeault cohomology, i.e. at the associated-graded for the Hodge filtration, over $[0]_{hod} \subset \mathcal{A}_{hod}$. A morphism $X_{Dol} \to V$ is said to be *semistable with vanishing Chern classes* if the morphism $X_{Dol} \to \pi_0(V)$ is constant, say with values at a point $t \in \pi_0(V)$, and if the morphism $X_{Dol} \to \tau_{\leq 1}(V)$ which becomes essentially

$$X_{Dol} \to K(\pi_1(V, t), 1),$$

is a semistable principal Higgs $\pi_1(V, t)$-torsor on X with vanishing rational Chern classes. One thing that we need to know here is that the condition of semistability with vanishing Chern classes is an open condition in the 1-stack of morphisms toward $\tau_{\leq 1} V$. This is Proposition 23 below.

If $\mathcal{V}$ is simply connected, i.e. the fundamental group object is trivial

$$\pi_1(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod}) = \{1\},$$

then the semistability condition is automatic. In this case we can apply Theorem 10.3.3 of [66] to get that $Tot^{W,F}(H_{DR})$ is a geometric n-stack over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$. For now we will restrict to the case where $\mathcal{V}$ is simply connected in our statement (Theorem 4) about $\mathcal{H}$.

Later on in Proposition 25 we show that if the fundamental group object of $\mathcal{V}$ is a flat linear group scheme then $Tot^{W,F}(H_{DR})$ is geometric; this leads to the statement of Theorem 6 which extends Theorem 4 to cover this case.

The total space of the weight filtration on H_{DR} has a formula similar to the previous one, namely:

$$Tot^{W}(H_{DR}) := \underline{Hom}\left(\frac{X_{DR} \times \mathcal{A}_{wt}}{\mathcal{A}_{wt}}, \frac{Tot^{W}(V_{DR})}{\mathcal{A}_{wt}}\right).$$

Next, we define similarly

$$H_{B,\mathbf{R}} := \underline{Hom}(X_{B,\mathbf{R}}, V_{B,\mathbf{R}}).$$

It is a real geometric n-stack cf [62] Proposition 2.1 and the comment on page 12, which refers to [59] Corollary 5.6; also in the present real case we use Lemma 10 above saying it suffices to check that $H_{B,\mathbf{C}}$ is geometric.

As before the weight filtration on $H_{B,\mathbf{R}}$ is defined by the formula

$$Tot^W(H_{B,\mathbf{R}}) := \underline{Hom}\left(\frac{X_{B,\mathbf{R}} \times \mathcal{A}_{wt}}{\mathcal{A}_{wt}}, \frac{Tot^W(V_{B,\mathbf{R}})}{\mathcal{A}_{wt}}\right).$$

This is geometric by [62] Proposition 2.1, [59] Corollary 5.6, and Lemma 10 above.

The analytic stacks associated to the algebraic n-stacks $Tot^W(H_{DR})$ and $Tot^W(H_{B,\mathbf{C}})$ are given again by the same formulae, namely:

$$Tot^W(H_{DR})^{\mathrm{an}} := \underline{Hom}\left(\frac{X_{DR}^{\mathrm{an}} \times \mathcal{A}_{wt}^{\mathrm{an}}}{\mathcal{A}_{wt}^{\mathrm{an}}}, \frac{Tot^W(V_{DR})^{\mathrm{an}}}{\mathcal{A}_{wt}^{\mathrm{an}}}\right)$$

and

$$Tot^W(H_{B,\mathbf{C}})^{\mathrm{an}} := \underline{Hom}\left(\frac{X_{B,\mathbf{C}}^{\mathrm{an}} \times \mathcal{A}_{wt}^{\mathrm{an}}}{\mathcal{A}_{wt}^{\mathrm{an}}}, \frac{Tot^W(V_{B,\mathbf{C}})^{\mathrm{an}}}{\mathcal{A}_{wt}^{\mathrm{an}}}\right).$$

Using these last, our map ζ_X gives rise to a map

$$\zeta_{\mathcal{H}} : Tot^W(H_{B,\mathbf{C}})^{\mathrm{an}} \to Tot^W(H_{DR})^{\mathrm{an}},$$

and the "GAGA theorem" of [59] says that this map is an equivalence. (Note that here we come across, for the first time, a case where the techniques of [59] are needed for nonabelian cohomology with coefficients in stacks where π_0 is nontrivial, in this case the π_0 is the base of the weight filtration, $\mathcal{A}_{wt}$.)

We have now constructed an object $\mathcal{H}$ in $PMH.nSTACK$, and if $\mathcal{V}$ is simply connected then $\mathcal{H}$ is in $PMH.nGEOM$. In Theorem 6 we will get $\mathcal{H}$ in $PMH.nGEOM$ if

$$\pi_i(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod})$$

is a flat linear group scheme over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$.

The case of coefficients in a pre-mhc

In order to treat the case of coefficients in a Dold-Puppe linearized object $\mathcal{V}$, we have to first treat the construction of an object $\underline{Hom}(X_M, \mathcal{C})$ when $\mathcal{C}$ is a split pre-mhc (this will then be applied to $\mathcal{C} = LGr^W(\mathcal{V})$). In fact it is obviously interesting to do this even when $\mathcal{C}$ is not split. Thus in this section we will define a pre-mhc $\underline{Hom}(X_M, \mathcal{C})$ whenever $\mathcal{C}$ is a pre-mhc.

We start with the following general construction. Suppose B is a base n-stack (of n-groupoids). Suppose $\mathcal{F} \to B$ is an n-stack over B and suppose that $\mathcal{C}$ is a complex of sheaves of abelian groups (which are assumed to be rational vector spaces) on B supported in $[-n, 0]$ (i.e. $\mathcal{C}$ is a morphism from B to the n-stack $nCPX$ of complexes of sheaves of abelian groups). Then we can define the relative morphism complex

$$\mathcal{M} = \underline{Hom}\left(\frac{\mathcal{F}}{B}, \frac{\mathcal{C}}{B}\right)$$

which is a complex of sheaves of abelian groups (again, rational vector spaces) over B. One way of doing this is to set $\mathcal{M}$ equal to the higher direct image complex of the restriction $\mathcal{C}|_{\mathcal{F}}$ via the morphism $p : \mathcal{F} \to B$, which is then truncated in positive degrees to give a complex supported in degrees $[-n, 0]$:

$$\mathcal{M} := \tau_{\leq 0}\mathbf{R}p_*(p^*\mathcal{C}).$$

Another way of doing it is to let $\mathcal{C}$ correspond to a relatively N-truncated $n + N$-stack $\tilde{\mathcal{C}}$ over B, to put

$$\tilde{\mathcal{M}}' := \underline{Hom}\left(\frac{\mathcal{F}}{B}, \frac{\tilde{\mathcal{C}}}{B}\right),$$

and to let

$$\tilde{\mathcal{M}} := \mathrm{fib}(\tilde{\mathcal{M}}' \to \tau_{\leq n}(\tilde{\mathcal{M}}')).$$

The fiber is taken over the zero-morphism as basepoint section, where the zero-morphism is the constant morphism with values the basepoint section of $\tilde{\mathcal{C}}$. Now $\tilde{\mathcal{M}}$ is a relatively N-connected $n + N$-stack over B so it corresponds to a complex of sheaves of abelian groups.

The above construction is compatible with the usual $\underline{Hom}$ of n-stacks via Dold-Puppe:

$$\underline{Hom}\left(\frac{\mathcal{F}}{B}, \frac{DP(\mathcal{C})}{B}\right) = DP\underline{Hom}\left(\frac{\mathcal{F}}{B}, \frac{\mathcal{C}}{B}\right).$$

The case which will interest us is when $\mathcal{F}$ is a formal category over B. See for example [66] for the definitions concerning this notion. Note that the

definitions of formal category and the various properties of such, were defined in [66] for the case where the base B was a scheme. These are immediately extended to the case where B is any n-stack by the usual trick of looking at a formal category over B (resp. one having certain properties) as a morphism from B into the 2-stack of formal categories (resp. the substack of those having the properties in question).

The following lemma is basically the same as Theorem 10.2.5 of [66], except that we have added the extension to the case of hemiperfect complexes.

Lemma 19 *Suppose $\mathcal{F}$ is a formal category over B which is projective and of smooth type relative to B. Suppose $\mathcal{C}$ is a hemiperfect complex over B. Then the morphism complex $\mathcal{M} = \underline{Hom}(\frac{\mathcal{F}}{B}, \frac{\mathcal{C}}{B})$ is a hemiperfect complex over B.*

Proof: To check that $\mathcal{M}$ is hemiperfect it suffices (by definition) to check that this is the case over any scheme $Z \to B$. Thus we may assume that B is a scheme, and in fact we may when necessary replace B by an open subset (part of an open cover). With this reduction we may assume that there is a perfect complex $\mathcal{K}$ supported in $[-n, 1]$ such that $\mathcal{C} = \tau_{\leq 0}\mathcal{K}$ (put $\mathcal{K}^i = \mathcal{C}^i$ for $i < 0$ and let $d : \mathcal{K}^0 \to \mathcal{K}^1$ be a morphism which has the vector scheme $\mathcal{C}^0$ as its kernel). Recall that p denotes the morphism $\mathcal{F} \to B$. We have

$$\mathcal{M} := \tau_{\leq 0}\mathbf{R}p_*(p^*\mathcal{C}) = \tau_{\leq 0}\mathbf{R}p_*(p^*\mathcal{K}).$$

Thus it suffices to prove that $\tau_{\leq 0}\mathbf{R}p_*(p^*\mathcal{K})$ is a perfect complex supported in $[-n, \infty)$. On the other hand, $\mathcal{K}$ is, up to quasiisomorphism, made up by taking successive cones of shifts of vector bundles. Perfection is preserved by cones, and higher direct image preserves triangles so up to quasiisomorphism it takes cones to cones. Shifts also commute with higher direct image and preserve perfection. Finally, by replacing B by an open cover we may assume that the vector bundles are trivial bundles, so we are reduced to claiming that

$$\mathbf{R}p_*(\mathcal{O})$$

is a perfect complex on B. Since $\mathcal{F}$ is of smooth type relative to B (i.e. the morphism $\mathcal{F} \to B$ is of smooth type in the terminology of [66] 8.6), its underlying scheme X is flat over B, and the higher direct image is calculated as the higher direct image of the "complex of differentials" for $\mathcal{F}$ (cf [66] 10.2.5):

$$\mathbf{R}p_*(\mathcal{O}) = \mathbf{R}p_{X/B,*}(\Omega^{\cdot}_{X/\mathcal{F}}).$$

The condition that $\mathcal{F}$ be of smooth type (cf [66]) means that $\Omega_{X/\mathcal{F}}^{\cdot}$ is a differential complex of locally free sheaves on X. Since X is by hypothesis projective over B, it is well-known that the higher direct image is a perfect complex. This completes the proof. |||

We have an analogue of the previous lemma designed to be applied to the "Betti" case.

Lemma 20 *Suppose $\mathcal{F}$ is an n-stack over B which is locally constant with fiber the homotopy type of a finite CW complex. Suppose $\mathcal{C}$ is a hemiperfect complex over B. Then we can define as previously a morphism complex $\mathcal{M} = \underline{Hom}(\frac{\mathcal{F}}{B}, \frac{\mathcal{C}}{B})$ which is hemiperfect over B.*

Proof: Left as an exercise. |||

Suppose $\mathcal{C}$ is a pre-mhc, and suppose X is a smooth projective variety. By using exactly the same formulae as in the previous construction starting on page 77 we can define a pre-mhc $\mathcal{H} := \underline{Hom}(X_M, \mathcal{C})$. In doing this translation, replace $\mathcal{V}$ by $\mathcal{C}$ (thus, V_{DR} by C_{DR} and so forth), and use the morphism complexes of Lemmas 19 and 20 in place of the usual internal $\underline{Hom}$ stacks used in the construction starting on page 77. We don't write out the formulae here because they are exactly the same as in the previous construction. Again the map ζ_X gives rise to the required equivalence $\zeta_{\mathcal{H}}$ between H_{DR} and $H_{B,\mathbf{C}}$. Note that in this case, analytic equivalences between hemiperfect complexes are algebraic so $\zeta_{\mathcal{H}}$ is actually an algebraic equivalence.

If $\mathcal{C}$ is a split pre-mhc then $\underline{Hom}(X_M, \mathcal{C})$ is also a split pre-mhc. Indeed, the action of $\mathbf{G}_m$ splitting the weight filtration on $\mathcal{C}$ induces an action on $\underline{Hom}(X_M, \mathcal{C})$ which again splits the weight filtration; this is because the object X_M is constant in the weight-filtration direction (i.e. in the direction of $\mathcal{A}_{wt}$) so it doesn't contribute to changing the degree in the weight filtration. Thus, $\underline{Hom}(X_M, \mathcal{C})$ is split as a pre-mhc.

The case of coefficients in a linearized pre-namhs

We now turn to the case which combines the two previous ones. Suppose $\mathcal{V}$ is a linearized pre-namhs. Recall that this means that $\mathcal{V}$ is a pre-namhs and that we have a split pre-mhc $LGr^W(\mathcal{V})$ whose Dold-Puppe is equivalent to $Gr^W(\mathcal{V})$ as split pre-namhs's. We now define a pre-namhs

$$\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$$

as follows. The underlying pre-namhs is the one which was constructed starting on page 77. We set

$$LGr^W(\mathcal{H}) := \underline{Hom}(X_M, LGr^W(\mathcal{V})),$$

which is a split pre-mhc. The above constructions are compatible with Dold-Puppe (i.e. the morphism complex defined in Lemmas 19 and 20 are compatible with the usual relative internal $\underline{Hom}$ stacks via Dold-Puppe), so we have a canonical equivalence

$$DP(LGr^W(\mathcal{H})) \cong Gr^W(\mathcal{H}).$$

This completes our construction of a linearized pre-namhs $\mathcal{H}$.

We sum up the results of this and the preceding paragraphs:

Theorem 4 *Suppose X is a smooth projective variety and $\mathcal{V}$ is a pre-namhs (resp. pre-mhc, resp. linearized pre-namhs). Suppose that $\mathcal{V}$ is simply connected (resp. no additional hypothesis in the pre-mhc case, resp. the same hypothesis in the linearized pre-namhs case). Then the object*

$$\underline{Hom}(X_M, \mathcal{V}) = \mathcal{H} = \{(H_{DR}, W, F), \ (H_{B,\mathbf{R}}, W), \ \zeta_{\mathcal{H}}\}$$

defined above is a pre-namhs (resp. pre-mhc, resp. linearized pre-namhs).

$$|||$$

The mixed Hodge complex conditions

The following proposition comes from Deligne's construction in "Hodge III" [14].

Theorem 5 *Suppose $\mathcal{C}$ is a truncated mixed Hodge complex (see page 58) and suppose that X is a smooth projective variety. Then the pre-mixed Hodge complex $\underline{Hom}(X_M, \mathcal{C})$ that we have constructed above is a truncated mixed Hodge complex, i.e. it satisfies conditions **Str** (p. 33), **MHC** (p. 55), **A1** (p. 31) and **A3** (p. 61).*

Proof: There is an actual mixed Hodge complex $\mathcal{D}$ (for example supported in $[-n, 1]$) such that $\mathcal{C}$ is the truncation of $\mathcal{D}$ into degrees ≤ 0. Then $\underline{Hom}(X_M, \mathcal{C})$ is the truncation of $\underline{Hom}(X_M, \mathcal{D})$. If we show that the latter is a truncated mixed Hodge complex, then by Corollary 11 its truncation will be a truncated mixed Hodge complex. Thus we are reduced to the case where $\mathcal{C}$ is a mixed Hodge complex. Furthermore one shows $(*)$, in fact, that the full cohomology complex of X_M with coefficients in $\mathcal{C}$ is a mixed Hodge complex supported in $[-n, \infty)$; the complex we denote $\underline{Hom}(X_M, \mathcal{C})$ is the truncation of this, so it is a truncated mixed Hodge complex.

To show $(*)$ that the cohomology is a mixed Hodge complex, it suffices to treat the associated-graded; but $Gr^W(\mathcal{C})$ decomposes as a direct sum of shifted split mixed Hodge structures. Thus it suffices to treat the case of cohomology with coefficients in a pure Hodge structure. It is well-known that it gives back a pure Hodge structure; the only problem is to verify that the shifts of weights work out appropriately. For this, we refer to the discussion on page 56. |||

Corollary 21 *If $\mathcal{C}$ is a split mixed Hodge complex (cf p. 54), and X is a smooth projective variety, then the pre-mhc $\underline{Hom}(X_M, \mathcal{C})$ is a split mixed Hodge complex.*

Proof: As remarked above, the splitting carries over to $\underline{Hom}(X_M, \mathcal{C})$. We just have to show that the truncated mixed Hodge complex is actually a mixed Hodge complex. The only direction in which the hemiperfect complexes involved in $\underline{Hom}(X_M, \mathcal{C})$ might not be perfect, is in the $\mathcal{A}_{wt}$ direction (in the $\mathcal{A}_{hod}$ direction they are perfect complexes because of the strictness conditon **Str**). However, the splitting of the weight filtration means that in the $\mathcal{A}_{wt}$ direction the complexes are constant so they are perfect complexes. Thus $\underline{Hom}(X_M, \mathcal{C})$ is a split mixed Hodge complex. |||

Corollary 22 *Suppose in the situation of Theorem 4 that $\mathcal{V}$ is a linearized pre-namhs and satisfies the strictness condition **Str** (Lemma 5 p. 33) and the mixed Hodge complex condition **MHC** (p. 55). Then the linearized pre-namhs $\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$ also satisfies the strictness condition **Str** and the condition **MHC**.*

Proof: These conditions are measured on the $LGr^W(\mathcal{V})$ and $LGr^W(\mathcal{H})$, so Theorem 5 and the previous corollary apply. |||

A couple of further remarks

The construction which to X and $\mathcal{V}$ associates $\mathcal{H}$ is functorial in $\mathcal{V}$ and contravariantly functorial in X (i.e. it gives an $n+1$-functor from the product of the category of smooth projective varieties with the $n + 1$-category of connected pre-namhs's (resp. connected linearized pre-namhs's), to the $n+1$-category of pre-namhs's (resp. linearized pre-namhs's)).

It is relatively clear from the above construction (namely, from the fact that X_M was taken to be constant in the $\mathcal{A}_{wt}$-direction) that $\mathcal{V} \mapsto \mathcal{H}$ depends only on the data of the *very presentable shape* of $X_{Hod}/\mathcal{A}_{hod}$ together with the very presentable shape of X_B and the equivalence between the Betti shape and the de Rham shape after composing with the analytification functor. In the future one hopes for a construction dealing with singular and open varieties, in which case there will surely be weight data contained in the appropriate generalization of X_M.

Semistable morphisms

In this section, we look more closely at $\underline{Hom}(X_M, \mathcal{V})$ when $\mathcal{V}$ is a pre-namhs which is not necessarily simply connected. This allows us to give a statement like Theorem 4 for the case of $\mathcal{V}$ not simply connected. The results of this section are not used in Part III.

For the purposes of this discussion and to reduce the volume of notation, we denote $B := \mathcal{A}_{wt} \times \mathcal{A}_{hod}$ and

$$\mathcal{F} := Tot^{W,F}(X_{DR}) = X_{Hod} \times \mathcal{A}_{wt} \to B$$

which is a formal category, projective and of smooth type over B. The important properties of this situation are:
(1)   There is a closed substack $B_1 \subset B$ such that

$$\mathcal{F}|_{B_1} = X_{Dol} \times B_1, \quad \mathcal{F}|_{B-B_1} = X_{DR} \times (B - B_1);$$

(2) For cohomology with vector sheaf coefficients, $\mathcal{F}/Z$ is of finite cohomological dimension; and
(3) There are only finitely many equivalence classes of $Spec(\mathbf{C})$-valued points in B.

We shall make use of these properties to get around what seem otherwise to be some delicate questions.

To continue with our notation, we fix a pre-namhs $\mathcal{V}$ and we let $V :=$ $Tot^{W,F}(V_{DR})$ denote the component which is a geometric n-stack over B. We assume that this is connected relative to B. Suppose Z is a scheme mapping to B. Let $\mathcal{F}_Z := \mathcal{F} \times_B Z$ and $V_Z := V \times_B Z$; by going to an etale neighborhood in Z if necessary, we may assume that there is a section $v : Z \to V_Z$. Now put $G_Z := \pi_1(V_Z/Z, v)$. This is a very presentable group sheaf over Z.

Hypothesis: That the fundamental group object of $\mathcal{V}$ is a flat linear group scheme; this means that for every scheme Z mapping to B and basepoint section $v : Z \to V_Z$ as above, the group sheaf $G_Z = \pi_1(V_Z/Z, v)$ is represented by a flat linear group scheme over Z.

(Note that there might not exist a basepoint of $\mathcal{V}$ over B so the fundamental group object doesn't exist *per se*; the condition on G_Z for every scheme Z is thus the definition of the above hypothesis.)

We recall the definition of the sub-n-stack "of semistable morphisms with vanishing Chern classes",

$$\underline{Hom}^{\mathrm{se},0}(\mathcal{F}/B, V/B) \subset \underline{Hom}(\mathcal{F}/B, V/B).$$

If $Z \xrightarrow{f} B$ is a scheme mapping to B, we shall say when a point $\varphi : \mathcal{F}_Z \to V_Z$ of the right hand n-stack, is a member of the substack. For the purposes of stating this condition we may replace Z by an etale open neigborhood and assume that V_Z admits a basepoint section v (the condition will be independent of the choice of basepoint section so the condition glues to give a condition over any Z).

Given a map φ as above, for any closed point $z \in Z$ we obtain a morphism $\varphi_z : \mathcal{F}_z \to V_z$. The first truncation in the Postnikov tower for V_z is

$$V_z \to \tau_{\leq 1}(V_z) = K(G_z, 1).$$

Thus φ_z projects to give a morphism $\rho_z : \mathcal{F}_z \to K(G_z, 1)$, i.e. a principal G_z-bundle over $\mathcal{F}_z$.

Recall from (1) above that there are two cases. If $f(z) \in (B - B_1)$ then $\mathcal{F}_z = X_{DR}$ and ρ_z is a principal G_z-bundle with flat connection over X. In this case the condition of being semistable with vanishing Chern classes is automatic, and we admit φ_z. If $f(z) \in B_1$ then $\mathcal{F}_z = X_{Dol}$ and ρ_z is a principal G_z-Higgs bundle over X. In this case we say that φ_z is "semistable with vanishing Chern classes" if the rational Chern classes of the principal

bundle ρ_z vanish for all invariant polynomials on G_z, and if for every linear representation E of G_z, the associated Higgs bundle $E(\rho_z)$ is semistable.

Now we say that a point $\varphi : \mathcal{F}_Z \to V_Z$ is "semistable with vanishing Chern classes" if for every point $z \in Z$ the restriction φ_z is semistable with vanishing Chern classes in the above sense. We define

$$\underline{Hom}^{\text{se},0}(\mathcal{F}/B, V/B)(Z) \subset \underline{Hom}(\mathcal{F}/B, V/B)(Z)$$

to be the full sub-n-groupoid of such points. This defines the substack.

Proposition 23 *Under the above notations for $\mathcal{F}/B$ and V/B and with the hypothesis that the G_Z are flat linear group schemes over Z, the sub-n-stack $\underline{Hom}^{\text{se},0}(\mathcal{F}/B, V/B)$ is an open substack of $\underline{Hom}(\mathcal{F}/B, V/B)$.*

Proof: Suppose $Z \to B$ is a morphism from a scheme, and suppose that $\varphi : \mathcal{F}_Z \to V_Z$ is a point. We have to show that the subset of points $z \in Z$ such that φ_z is semistable with vanishing Chern classes, is an open subset of Z. For this we can localize on Z so we assume that there is a basepoint section v of V_Z and let G_Z be the fundamental group as above. Let ρ be the principal G_Z-bundle over $\mathcal{F}_Z$ obtained by the first Postnikov projection of φ. We have to prove that the set of points $z \in Z$ such that ρ_z is semistable with vanishing Chern classes, is open. Let Z_1 be the inverse image of B_1; it is a closed subscheme of Z and outside of Z_1 all of the points are semistable with vanishing Chern classes. The complement of the set in question is thus contained in Z_1; since Z_1 is closed in Z, it suffices to prove that this complement is closed in Z_1, in other words it suffices to prove that the subset of points $z \in Z_1$ where ρ_z is semistable with vanishing Chern classes, is open in Z_1. We are reduced to this question on Z_1 so we may assume $Z = Z_1$. With this assumption $\mathcal{F}_Z = X_{Dol} \times Z$ and ρ is a principal Higgs bundle with structure group G_Z.

When G_Z is the constant sheaf of groups $GL(n)$ over Z, the openness of the set of points which are semistable with vanishing Chern classes, is well-known. The difficulty here is to extend this to the case of structure group sheaf G_Z which is flat and linear but not necessarily constant.

Let $G_Z \to GL(E/Z)$ be the faithful representation given by the condition that G_Z is linear. Let $E(\rho)$ be the associated Higgs bundle on $X_{Dol} \times Z/Z$. If $z \in Z$ is a point then we obtain the Higgs bundle $E(\rho)_z$ which is associated to the principal Higgs G_z-bundle ρ_z by the faithful representation $G_z \to GL(E_z)$. Recall that ρ_z is said to be semistable if the Higgs bundles associated

to *every* representation of G_z are semistable; however, recall from [57] p. 86 that if ρ_z has vanishing Chern classes, then it suffices to check this for one faithful representation of G_z. Therefore the set of points z such that ρ_z is semistable with vanishing Chern classes, is the same as the set of points where ρ_z has vanishing Chern classes and where $E(\rho)_z$ is semistable. The set of points where $E(\rho)_z$ is semistable, is open cf [58]. Thus we may work inside this set, so we may assume from now on that $E(\rho)_z$ is semistable for all $z \in Z$.

Let $P \to X \times Z$ be the geometric principal G_Z-bundle underlying the principal Higgs bundle ρ. We have to prove that the set of points $z \in Z$ where $P_z = P|_{X \times \{z\}}$ has vanishing Chern classes, is open.

It is possible to define the Chern classes in de Rham cohomology in an algebraic way, so the set of points where P_z has vanishing Chern classes is a constructible set. To prove openness it suffices therefore to prove that it is open in the usual topology on Z (recall that we work over $Spec(\mathbf{C})$). So chose a point z such that P_z has vanishing Chern classes. In view of the semistability hypothesis, we obtain that the principal Higgs bundle ρ_z corresponds to a flat principal G_z-bundle via the correspondence between Higgs bundles and local systems [57]. This correspondence preserves the C^∞ type of the bundle, so we get that the C^∞ principal G_z-bundle on X underlying ρ_z, i.e. P_z, is flat. By Deligne and Sullivan [12], there is a finite etale covering $X' \to X$ such that $P_z|_{X'}$ is a trivial principal bundle, i.e. has a C^∞ section s_z.

Here is where we use the flatness hypothesis on G_Z. In characteristic zero flatness implies smoothness for group schemes, so G_Z is smooth over Z. Since P is modelled locally on G_Z, we obtain that the morphism $P \to X \times Z$ is smooth. Let Z' be a neighborhood of z in the usual topology in Z, such that there exists a retraction of Z' to $\{z\}$ where the trajectories are smooth curves in Z'. Assume that the retraction is obtained by following a vector field along these curves; and assume that over Z' we have lifted the vector field to a vector field on P (projecting to zero in the direction X of the base $X \times Z$). Such a neighborhood exists.

Let P' be the pullback of P to $X' \times Z'$. Following our lifts of vector fields on P' gives a trivialization of the underlying differential manifold, $P' \cong (P_z \times_X X') \times Z'$. Via this trivialization, we can extend our section s_z to a section s of the bundle P'. In particular, for every point $y \in Z'$ the C^∞ principal G_y-bundle P'_y on X' is trivial. In particular it has vanishing Chern classes in rational cohomology. The morphism $H^i(X, \mathbf{Q}) \to H^i(X', \mathbf{Q})$ is

injective, so we obtain that for every point $y \in Z'$ the principal G_y-bundle P_y has vanishing Chern classes on X. Thus Z' provides the neighborhood of z necessary to show that the set of points where P_y has vanishing Chern classes, is open. |||

Counterexample: The hypothesis of flatness (i.e. smoothness) of G_Z is essential in the above argument; here is a counterexample showing that we cannot remove it. Fix a stable vector bundle E_0 with vanishing Chern classes for structure group $GL(n)$ over a curve X. The bundle E has a Borel reduction, i.e. a flag of sub-line-bundles, such that the degrees of the line bundles are nonzero. Let B denote the Borel subgroup which is the structure group for this reduction. Let $Z = \mathbf{A}^1$ and let G_Z be a group scheme over Z, contained in $GL(n) \times Z$, with generic fiber B (over every point except the origin) and special fiber $GL(n)$ over the origin. Note that G_Z isn't flat. Let E be the bundle $E_0 \times Z$ over $X \times Z$, but considered as a principal G_Z-bundle using the B-reduction for E_0 outside the origin. The fiber over the origin is the principal $GL(n)$-bundle E_0 which we started with; it is semistable with vanishing Chern classes. However for any other point $z \neq 0$ the principal $G_z = B$-bundle E_z doesn't have vanishing Chern classes (it isn't semistable either). Thus the set of points of Z where the bundle is semistable with vanishing Chern classes, is reduced to $\{0\}$ which isn't open.

Next we turn to the problem of local geometricity. This part of the discussion doesn't use semistability or our special situation for $\mathcal{F}/B$; however we do use the hypothesis that the fundamental group sheaf is a flat linear group scheme.

Proposition 24 *Suppose $\mathcal{F}/B$ is a formal category which is projective and of smooth type over a base n-stack B, with finite cohomological dimension for vector sheaf coefficients. Suppose that V/B is a relatively connected geometric n-stack such that for any scheme $Z \to B$ with basepoint section $v : Z \to V_Z$, the fundamental group sheaf $G_Z = \pi_1(V_Z/Z, v)$ is represented by a flat linear group scheme. Then $\underline{Hom}(\mathcal{F}/B, V/B)$ is a locally geometric n-stack.*

Proof: Suppose $Z \to B$ is a morphism from a scheme. By going to an etale neighborhood we may assume that V_Z has a basepoint section v, so we obtain the group sheaf G_Z (which by hypothesis is a flat linear group scheme over Z) and the first stage in the Postnikov tower for V_Z is $K(G_Z/Z, 1)$. Let

$$M_Z := \underline{Hom}(\mathcal{F}_Z/Z, K(G_Z/Z, 1)/Z),$$

which is the moduli 1-stack for principal G_Z-bundles over $\mathcal{F}_Z$. By [66], Proposition 10.4.4, M_Z is locally geometric (i.e. it is an Artin algebraic 1-stack which is locally of finite type). We will show that the morphism

$$p : \underline{Hom}(\mathcal{F}/B, V/B) \times_B Z \to M_Z$$

is geometric, yielding the desired conclusion. For this, we may suppose that $\rho_Z : Z \to M_Z$ is a point, and it suffices to show that the fiber of p over ρ_Z is a geometric n-stack over Z. The point ρ_Z may also be considered as a morphism $\rho_Z : \mathcal{F}_Z \to K(G_Z/Z, 1)$ or equivalently a principal G_Z-bundle over $\mathcal{F}_Z$. Let U_Z be the fiber in the fibration sequence

$$U_Z \to V_Z \to K(G_Z/Z, 1).$$

It is a simply connected geometric n-stack with action of the group G_Z. Twisting this n-stack by the principal G_Z-bundle ρ_Z we obtain a relatively 1-connected geometric n-stack

$$U_Z(\rho_Z) \to \mathcal{F}_Z.$$

The fiber of p over ρ is the relative section stack

$$\Gamma(\mathcal{F}_Z/Z, U_Z(\rho_Z)).$$

We have to prove that this is a geometric n-stack over Z. This is a twisted version of the statement 10.3.3 of [66], and we follow the idea of the proof that was sketched there.

The considerations we discuss here are fundamentally well-known to algebraic topologists.

In order to write the proof in an efficient way, we work with ∞-stacks (which we assume without further stating this to be stacks of ∞-groupoids or equivalently simplicial presheaves). Suppose $f : R \to Z$ is a simply connected ∞-stack over a base scheme Z. We say that R is *cohomologically geometric* if $\mathbf{R}f_*(\mathcal{O})$ is a perfect complex (supported in $[0, \infty)$) over Z. We say that R is *residually geometric* if for any N the relative truncation $\tau_{\leq N}(R/Z)$ is an n-almost geometric n-stack in the terminology of [66] §7.3. Recall the result 7.3.5 of [66] which, in terms of our present definitions, implies that if R is cohomologically geometric then it is residually geometric. (The converse is probably also true but we don't have a proof.) Recall also [66] 7.3.1 saying

that if R is n-truncated relative to Z and if R is residually geometric then R is in fact geometric.

We note the following canonical construction (this construction is the "Goodwillie tower of the identity functor" see [21]). Define

$$\Sigma_0 := \Omega^\infty S^\infty (R/Z)$$

to be the Dold-Puppe of the dual $\mathbf{R}f_*(\mathcal{O})^*$. There is a natural map

$$R \to \Omega^\infty S^\infty (R/Z) = \Sigma_0.$$

If R is cohomologically geometric then Σ_0 is the Dold-Puppe of a perfect complex so it again is cohomologically geometric. Let R_1 be the fiber of the above map. It fits into a fibration sequence

$$\Omega\Sigma_0 \to T \to R,$$

so an easy spectral sequence argument shows that T is also cohomologically geometric. Iterating this construction we obtain a sequence of cohomologically geometric ∞-stacks R_i with fibration sequences

$$R_{i+1} \to R_i \to \Sigma_i$$

where the Σ_i are the Dold-Puppe of perfect complexes. Furthermore the R_i eventually become more and more connected, i.e. there is a sequence $k_i \to \infty$ such that R_i are k_i-connected.

The above constructions relative to a base scheme are canonical (with canonical homotopy coherence at all orders) so they relativize: if B is any ∞-stack and if R/B is an ∞-stack which is cohomologically geometric (i.e. for any scheme $Z \to B$ the pullback $R_Z = R \times_B Z$ is cohomologically geometric over Z) then we obtain a sequence of ∞-stacks R_i/B fitting into fibration sequences as above, with Σ_i again being the relative Dold-Puppe of perfect complexes relative to B, the R_i are cohomologically geometric, and they are more and more connected relative to B.

We now apply this construction to the study of relative section stacks. Suppose $\mathcal{F}_Z \to Z$ is a morphism and suppose R is a cohomologically geometric ∞-stack over $\mathcal{F}_Z$. We obtain a morphism

$$\Gamma(\mathcal{F}_Z/Z, R) \to \Gamma(\mathcal{F}_Z/Z, \Sigma_0).$$

We will treat below the case of sections of the Dold-Puppe of a perfect complex; assume for now that these are known to be residually geometric. Thus $\Gamma(\mathcal{F}_Z/Z, \Sigma_0)$ is residually geometric. The fiber of the above morphism over any section of $\Gamma(\mathcal{F}_Z/Z, \Sigma_0)$ is of the form $\Gamma(\mathcal{F}_Z/Z, R_1')$ for an ∞-stack R_1' over $\mathcal{F}_Z$ which, over any scheme mapping to $\mathcal{F}_Z$, locally looks like the fiber R_1 defined above. Thus the fiber R_1' is cohomologically geometric relative to $\mathcal{F}_Z$ and we can again iterate the construction. We assumed that $\mathcal{F}_Z$ has finite cohomological dimension relative to Z for vector sheaf coefficients. From this assumption, for any fixed N the later terms in the iteration eventually become irrelevant to $\tau_{\leq N}\Gamma(\mathcal{F}_Z/Z, R)$, and up to these later terms we have expressed $\Gamma(\mathcal{F}_Z/Z, R)$ as the result of a process of fibrations where the base is always a residually geometric ∞-stack (this remains to be treated below), and the fiber is again decomposed further in the same way. If the base and fibers are residually geometric then the total space is residually geometric (this is stated with only an indication of the proof in [66] Lemma 7.3.3 but we leave it as an exercise here too). Finally we obtain that $\tau_{\leq N}\Gamma(\mathcal{F}_Z/Z, R)$ is a truncation of a residually geometric ∞-stack, thus it is an N-almost geometric N-stack.

If the original R was n-truncated relative to $\mathcal{F}_Z$ i.e. if it was an n-stack relative to $\mathcal{F}_Z$ then choosing $N \geq n+1$ and applying [66] Lemma 7.3.1, we obtain finally that

$$\Gamma(\mathcal{F}_Z/Z, R) = \tau_{\leq N}\Gamma(\mathcal{F}_Z/Z, R)$$

is a geometric n-stack relative to Z. This applies to our twisted $U_Z(\rho_Z)$ to yield that

$$\Gamma(\mathcal{F}_Z/Z, U_Z(\rho_Z))$$

is a geometric n-stack over Z.

To complete the proof we have to treat the case of coefficients in a perfect complex. If C is a perfect complex over $\mathcal{F}_Z$ supported in $(-\infty, 0]$ and if $\Sigma = DP(C/\mathcal{F}_Z)$ then

$$\Gamma(\mathcal{F}_Z/Z, \Sigma) = DP\tau^{\leq 0}\Gamma(\mathcal{F}_Z/Z, C).$$

Thus, if we can show that $\Gamma(\mathcal{F}_Z/Z, C)$ is a perfect complex (supported in $(-\infty, \infty)$) then we obtain that $\Gamma(\mathcal{F}_Z/Z, \Sigma)$ is residually geometric as desired. In fact it suffices to show that $\Gamma(\mathcal{F}_Z/Z, C)$ is *residually perfect* cf [66] Definition 7.3.4.

Now $\Gamma(\mathcal{F}_Z/Z, C)$ is calculated by the relative de Rham complex for $\mathcal{F}_Z/Z$, twisted by C. There is a spectral sequence whose starting terms are the cohomology of X/Z with coefficients in C tensored with the vector bundles appearing in the de Rham complex for $\mathcal{F}_Z$ (here X denotes the scheme underlying the formal category $\mathcal{F}_Z$). Thus, in order to obtain the desired result it suffices to consider a flat morphism of schemes $g : X \to Z$ and to prove that if D is a perfect complex on X supported in $(-\infty, m]$ then the direct image $\Gamma(X/Z, D) = \mathbf{R}g_*(D)$ is a residually perfect complex on Z. The residually perfect condition means that for any N the truncation $\tau^{[-N,N]}\mathbf{R}g_*(D)$ into the interval $[-N, N]$ should be quasiisomorphic to the truncation of a perfect complex. To prove this suppose k is the cohomological dimension of X/Z and choose an explicit resolution L of D by vector bundles in the interval $[-N-k, m]$. In other words choose a complex of vector bundles L (supported in $[-N-k-1, m]$ say) with a morphism $L \to D$ which is exact in the interval $[-N-k, m]$. This is possible since X/Z is flat and projective. Now

$$\tau^{[-N,N]}\mathbf{R}g_*(L) \to \tau^{[-N,N]}\mathbf{R}g_*(D)$$

is a quasiisomorphism. Since L is a complex of vector bundles over X, the well-known fact that the higher direct image of a vector bundle under a flat morphism is a perfect complex, implies that $\tau^{[-N,N]}\mathbf{R}g_*(D)$ is the truncation of a perfect complex. This being true for any N we obtain that $\mathbf{R}g_*(D)$ is a residually perfect complex. This completes the last part of the proof. $|||$

Finally we go back to our original situation and use the openness of semistability as well as condition (3) to get a geometric n-stack of semistable morphisms.

Proposition 25 *Suppose $\mathcal{F}/B$ is a formal category which is projective and of smooth type over a base 1-stack B. Suppose that we are in the situation of (1)-(3) above (with B geometric). Suppose V/B is a geometric n-stack which is relatively connected, and whose relative fundamental group object is a flat linear group scheme (as above this condition is made relative to every scheme $Z \to B$ such that a basepoint section v of V_Z exists). Then $\underline{Hom}^{se,0}(\mathcal{F}/B, V/B)$ is a geometric n-stack over B.*

Proof: By Proposition 24, the n-stack $\underline{Hom}(\mathcal{F}/B, V/B)$ is locally geometric. Furthermore, the proof of that proposition actually gives the stronger statement that if $\tau_{\leq 1}(V/B)$ denotes the first stage in the Postnikov tower,

then
$$\underline{Hom}(\mathcal{F}/B, \tau_{\leq 1}(V/B)/B)$$

is locally geometric, and the morphism

$$\underline{Hom}(\mathcal{F}/B, V/B) \to \underline{Hom}(\mathcal{F}/B, \tau_{\leq 1}(V/B)/B)$$

is geometric. Proposition 23 says that

$$\underline{Hom}^{\mathrm{se},0}(\mathcal{F}/B, V/B) \subset \underline{Hom}(\mathcal{F}/B, V/B)$$

is an open substack; thus it too is locally geometric. Furthermore it is pulled back from the open substack

$$M := \underline{Hom}^{\mathrm{se},0}(\mathcal{F}/B, \tau_{\leq 1}(V/B)/B) \subset \underline{Hom}(\mathcal{F}/B, \tau_{\leq 1}(V/B)/B).$$

Thus we have a geometric morphism

$$\underline{Hom}^{\mathrm{se},0}(\mathcal{F}/B, V/B) \to \underline{Hom}(\mathcal{F}/B, \tau_{\leq 1}(V/B)/B) = M$$

between locally geometric n-stacks, and it suffices to prove that the target M of this morphism is geometric, i.e. of finite type. The reader may be happy to note that M is only a 1-stack—thus it is an Artin algebraic stack locally of finite type over B and we would like to say that it is of finite type over B.

We know that over every point $b \in B$, the fiber M_b (which is just a moduli stack of semistable principal bundles with vanishing Chern classes on X_{Dol} or X_{DR}) is of finite type. Normally, this type of pointwise information for each fiber isn't sufficient to conclude that the total stack is of finite type over the base. However, we have the hypothesis (3) that there are only finitely many equivalence classes of points in the base B, and this allows us to obtain the desired conclusion.

Let $Z = \coprod_{i \in I} Z_i$ be a disjoint union of schemes Z_i of finite type, with a smooth surjection

$$Z \to M.$$

Let $B(Spec(\mathbf{C}))$ denote the finite set of points of B, and for each $b \in B(Spec(\mathbf{C}))$ let M_b denote the preimage in M. We know (see [58]) that each M_b is of finite type, since it is a moduli stack of semistable principal Higgs G_b-bundles of vanishing Chern classes on X (if $b \in B_1$) or of principal G_b-bundles with flat connection (if $b \in B - B_1$). In particular, there is a finite subset of indices $I' \subset I$ such that for any point b, M_b is covered by

the images of the X_i for $i \in I'$. Let $X' := \coprod_{i \in I'} X_i$ (it is a scheme of finite type). The map $X' \to M$ is smooth, and for any $Spec(\mathbf{C})$-valued point $m \in M(Spec(\mathbf{C}))$ we have that the preimage of m in X' is nonempty. Thus, for any scheme Y mapping to M, the map $X' \times_M Y \to Y$ has is smooth with the same pointwise surjectivity property; but $X' \times_M Y$ is a geometric n-stack so this means that the map $X' \times_M Y \to Y$ is a smooth surjection. Therefore $X' \to M$ is a smooth surjection, and M is of finite type. This completes the proof of the proposition. |||

As a corollary we obtain the following statement.

Theorem 6 *Suppose $\mathcal{V}$ is a pre-namhs (resp. linearized pre-namhs) which is connected, and such that the fundamental group objects such as*

$$\pi_1(Tot^{W,F}(V_{DR})/\mathcal{A}_{wt} \times \mathcal{A}_{hod})$$

are flat linear group schemes. Suppose X is a smooth projective variety. Then the object $\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$ is a pre-namhs (resp. linearized pre-namhs).

Proof: In the construction of $\mathcal{H}$ given at the start of this chapter, the only open question was whether $Tot^{W,F}(H_{DR})$ was a geometric n-stack over $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$. Recall that

$$Tot^{W,F}(H_{DR}) := \underline{Hom}^{se,0}\left(\frac{X_{Hod} \times \mathcal{A}_{wt}}{\mathcal{A}_{wt} \times \mathcal{A}_{hod}}, \frac{Tot^{W,F}(V_{DR})}{\mathcal{A}_{wt} \times \mathcal{A}_{hod}}\right).$$

The hypothesis of the theorem says exactly that the hypothesis of Proposition 25 applies, so that proposition gives geometricity of $Tot^{W,F}(H_{DR})$. The rest of the construction of $\mathcal{H}$, including the treatment of the linearization, is as indicated at the start of the chapter. |||

The basic conjectures

We can now formulate our basic conjectures concerning nonabelian mixed Hodge structures. These three conjectures 1, 2 and 3 are all statements we expect that one could prove within a medium-range amount of time. We decided to post the current version of the paper as is, without proofs of these conjectures, because it seems clear (e.g. from the difficulty which we already encountered just treating the mixed Hodge structures on the homotopy groups $\pi_i(\mathcal{V})$) that their proofs will require a significant amount of technical material and it didn't seem necessary to burden the reader—or ourselves—with that at present.

Conjecture 1 *Suppose X is a connected smooth projective variety. Suppose $\mathcal{V}$ is a nonabelian mixed Hodge structure. Then the linearized pre-namhs $\mathcal{H} = \underline{Hom}(X_M, \mathcal{V})$ is a nonabelian mixed Hodge structure.*

Some of the main properties (**Str** and **MHC** above) are given by Proposition 22, at least under a reasonable hypothesis on $\mathcal{V}$. The main things which remain to be shown are: (I) flatness of the Hodge filtration (condition **Fl**) and the conditions **A1** and **A2** on the annihilator ideals; and (III) the third Hodge-theoretic condition **A3** on the annihilator ideals.

A subsidiary conjecture is the following one.

Conjecture 2 *Suppose $\mathcal{V}$ and $\mathcal{Y}$ are nonabelian mixed Hodge structures, and suppose that $\mathcal{Y}$ is 1-connected (i.e. its π_0 and π_1 are trivial). Then there is a nonabelian mixed Hodge structure $\underline{Hom}(\mathcal{Y}, \mathcal{V})$ (with natural choices for the Hodge and weight filtrations), again functorial in $\mathcal{V}$ and (contravariantly) in $\mathcal{Y}$.*

Denote by $*$ the trivial nonabelian mixed Hodge structure where all objects are a single point (objects relative to a base such as $\mathcal{A}_{wt} \times \mathcal{A}_{hod}$ are equal to that base). If $\mathcal{V}$ is a namhs then a morphism $* \to \mathcal{V}$ is the nonabelian analogue of a Hodge class of type $(0,0)$ in $\mathcal{V}$. Thus we shall call such a point a *Hodge class in $\mathcal{V}$*.

The above internal $\underline{Hom}$ should be compatible with the notion of morphism, in that a morphism from $\mathcal{Y}$ to $\mathcal{V}$ should be the same thing as a Hodge class in $\underline{Hom}(\mathcal{Y}, \mathcal{V})$. Getting back to the first conjecture, in cases where it holds we can define a *morphism from X_M to $\mathcal{V}$* as being a morphism $* \to \underline{Hom}(X_M, \mathcal{V})$.

We now state another conjecture about representability in the simply connected case. This conjecture (if it turns out to be true) states how to define the "mixed Hodge structure on the homotopy type of X".

Conjecture 3 *Suppose X is a simply connected smooth projective variety. Then there is a* universal morphism *to a simply connected nonabelian mixed Hodge structure*

$$X_M \to \mathcal{Y} = \mathcal{MHS}(X)$$

with the property that for any nonabelian mixed Hodge structure $\mathcal{V}$ the resulting morphism

$$\underline{Hom}(\mathcal{Y}, \mathcal{V}) \to \underline{Hom}(X_M, \mathcal{V})$$

is an equivalence. Furthermore this representing object specializes to the n-stack representing the de Rham (resp. Betti) shape of X (cf [66]), i.e.
Y_{DR} is the n-stack representing the very presentable shape of X_{DR};
$Tot^F(Y_{DR})$ is the Hodge filtration on the de Rham representing object, as constructed in [66]; and
$Y_{B,\mathbf{C}}$ is the n-stack representing the very presentable shape of X_B.
Finally, the mixed Hodge structures on the homotopy vector spaces $\pi_i^{\mathrm{nu}}(\mathcal{Y}) = \pi_i(X_B) \otimes \mathbf{C}$ coincide with those defined by Morgan and Hain.

It is a consequence of the functoriality in the first two conjectures and the universality in this last conjecture that $\mathcal{MHS}(X)$ would be functorial in X. In other words, if $X \to Z$ is a morphism of smooth projective varieties then we would get a morphism of nonabelian mixed Hodge structures

$$\mathcal{MHS}(X) \to \mathcal{MHS}(Z)$$

or equivalently a Hodge class in $\underline{Hom}(X_M, \mathcal{MHS}(Z))$. It is natural to ask which morphisms of nonabelian mixed Hodge structures come from morphisms of varieties. This is a subtle question which could be viewed as a *nonabelian analogue of the Hodge conjecture*:
—do all Hodge classes in $\underline{Hom}(X_M, \mathcal{MHS}(Z))$ come from morphisms of varieties $X \to Z$?

In the non-simply connected case, there cannot exist a universal map such as given by Conjecture 3. Nonetheless, one might try to get back from the nonabelian cohomology to a mixed Hodge structure on the "higher Malcev completion" such as was proposed at the end of [64].

Finally we state a conjecture which is intended to show how the condition **A1** fits in with the result of Deligne-Griffiths-Morgan-Sullivan [11] that smooth projective varieties are formal.

Conjecture 4 *Suppose X is a simply connected finite CW complex, and let X_B denote the constant ∞-stack with values the Poincaré ∞-groupoid of X. Then the following two properties are equivalent:*
(1) for all n and for any 1-connected namhs $\mathcal{V}$, the pre-weight filtered n-stack

$$\underline{Hom}(X_B, (V_{B,\mathbf{C}}, W))$$

*satisfies condition **A1** (cf page 31);*
(2) the dga associated to X by the theory of Sullivan and Morgan, is formal.

Some vague calculations with dga's seem to support this conjecture.

Recall in a similar vein the result of the exercise on p. 40, saying that if Y is a quadratic cone at a point y then $(Y, W(y))$ satisfies **A1**. One could also conjecture that that result is "if and only if", and extend the above conjecture to the non-simply connected case. We hope to treat this type of question in a future paper.

If these statements turn out to be substantially true, then one could say that the properties of formality; of degeneration of spectral sequences at E_2; and the property **A1** are all sort of the same thing.

Variations of nonabelian mixed Hodge structure

Suppose S is a smooth base scheme. We will sketch a definition of *variation of nonabelian mixed Hodge structure (vnamhs) over S*. The basic idea is to combine the notion of Griffiths transversality for the Hodge filtration that was defined using the object S_{Hod} in [60], with the weight filtration as we are defining it in the present paper.

Most aspects of what we say in this section will be highly conjectural. Nonetheless we feel that the reader might legitimately be interested to know about a likely direction of future development.

Recall that $S_{Hod} \to \mathcal{A}_{hod}$ is a formal category whose fiber over [1] is S_{DR} and whose fiber over [0] is S_{Dol} (recall that starting from the $\mathbf{G}_m$-equivariant formal category over $\mathbf{A}^1_{hod}$ defined in [60] [66], we took the quotient object by $\mathbf{G}_m$ to get a formal category over $\mathcal{A}_{hod}$ cf p. 76). Let OBJ be one of the $n+1$-stacks we consider in the present paper, and look at objects in OBJ. An object $\underline{\mathcal{U}} \to S_{Hod}$, which may also be seen as a functor

$$\underline{\mathcal{U}} : S_{Hod} \to OBJ,$$

restricts over S_{DR} to an object $\underline{U}_{DR}$ (or $\underline{U}_{DR} : S_{DR} \to OBJ$), which is the same thing as an object over S together with a flat connection. On the other hand, for every point $s \in S$ we obtain a morphism

$$\mathcal{A}_{hod} = \{s\}_{Hod} \to S_{Hod},$$

so the restriction of $\mathcal{U}$ to $\{s\}_{Hod}$ gives a filtration $F(s)$ of the fiber $U_{DR}(s)$. The fact that $\mathcal{U}$ is a family over S_{Hod} encapsulates *Griffiths transversality* for this family of filtrations together with the connection. This is explained in somewhat more detail in [60].

Apply the above discussion to the case where $OBJ = F^{DP}.nGEOM$ is the $n + 1$-stack of pre-weight-filtered geometric n-stacks. An object

$$S_{Hod} \rightarrow F^{DP}.nGEOM$$

gives a family of linearized weight filtrations with a flat connection over S, and with a varying family of Hodge filtrations (including Hodge filtrations on the perfect complexes of the linearizations) such that the Hodge filtrations satisfy Griffiths transversality for the weight filtration. Denote the above morphism by

$$(\underline{U}_{Hod}, W, LGr^W) : S_{Hod} \rightarrow F^{DP}.nGEOM.$$

The restriction over S_{DR} is denoted $(\underline{U}_{DR}, W, LGr^W)$.

Note that the "Hodge filtration" which we would have called F, is integrated into the object U_{Hod}. Thus it is reasonable to think of the object $\underline{U}_{Hod}$ as being $(\underline{U}_{DR}, F)$. On a technical level, however, F is not a filtration of the object $\underline{U}_{DR}$ as this would imply that F were flat with respect to the connection. The object $\underline{U}_{Hod}$ represents a family of filtrations $F(s)$ on the fibers $\underline{U}_{DR}(s)$, satisfying Griffiths transversality with respect to the connection. We will however sometimes use the abuse of notation and call this $(\underline{U}_{DR}, F)$. This takes care of the part $(\underline{U}_{DR}, W, F)$ together with its linearization L, in the notion of vnamhs.

For the notion of *pre-vnamhs* we will want to drop the existence of a linearization, in this case we just look at a morphism

$$(\underline{U}_{Hod}, W) : S_{Hod} \rightarrow F.nGEOM.$$

If $s \in S$ is a point, then the restriction of $(\underline{U}_{Hod}, W)$ to $\{s\}_{Hod} \cong \mathcal{A}_{hod}$ becomes an object

$$\mathcal{A}_{hod} \rightarrow F.nGEOM,$$

i.e. an object of $F.F.nGEOM$. This is just the part which was denoted by $(\underline{U}_{DR}(s), F(s), W(s))$ in the notion of pre-namhs. In the linearized case we get the same plus a linearization $(LGr^{W(s)}(\underline{U}_{DR}(s)), F(s))$.

We next look at the "Betti" part of the definition. Recall again that $S_{B,\mathbf{R}}$ is the constant stack with values the Poincaré groupoid of S^{top}. Since this will be used as the base for a family of n-stacks, we need to go to the $n + 1$-truncation of S^{top}, in other words we let $S_{B,\mathbf{R}}$ denote the constant $n + 1$-stack

on the real site, whose values are $\Pi_{n+1}(S^{\mathrm{top}})$. Now the piece $(\underline{U}_{B,\mathbf{R}}, W, L)$ in the notion of vnamhs, corresponds to a functor of $n+1$-stacks

$$(\underline{U}_{B,\mathbf{R}}, W, LGr^W) : S_{B,\mathbf{R}} \to F^{DP}.nGEOM.$$

This should be seen as a local system over S^{top} of linearized filtered objects. Again, for the notion of pre-vnamhs we will drop the existence of the linearization and just look at

$$(\underline{U}_{B,\mathbf{R}}, W) : S_{B,\mathbf{R}} \to F.nGEOM.$$

If $s \in S$ is a point then $\{s\}_{B,\mathbf{R}}$ is just $*$ (i.e. $Spec(\mathbf{R})$) and the restriction of our functor to $\{s\}_{B,\mathbf{R}}$ is just a real filtered geometric n-stack $(\underline{U}_{B,\mathbf{R}}(s), W(s))$.

Finally, recall that we have a morphism

$$\zeta_S : S^{\mathrm{an}}_{DR} \to S^{\mathrm{an}}_{B,\mathbf{C}}.$$

Thus we can ask to have an equivalence (between pre-weight filtered n-stacks over S^{an}_{DR}) of the form

$$\zeta_{\underline{\mathcal{U}}} : (\underline{U}_{DR}, W, LGr^W)^{\mathrm{an}} \cong \zeta_S^*(\underline{U}_{B,\mathbf{R}}, W, LGr^W)^{\mathrm{an}}.$$

As above, for the notion of pre-vnamhs we will drop the linearization.

We have now indicated all of the elements going into the notion of a *pre-vnamhs $\underline{\mathcal{U}}$* over S: this consists of morphisms

$$(\underline{U}_{Hod}, W) : S_{Hod} \to F.nGEOM$$

and

$$(\underline{U}_{B,\mathbf{R}}, W) : S_{B,\mathbf{R}} \to F.nGEOM,$$

together with an analytic equivalence of filtered n-stacks over S^{an}_{DR},

$$\zeta_{\underline{\mathcal{U}}} : (\underline{U}_{DR}, W)^{\mathrm{an}} \cong \zeta_S^*(\underline{U}_{B,\mathbf{R}}, W)^{\mathrm{an}}.$$

A *linearized pre-vnamhs $\underline{\mathcal{U}}$* is a pre-vnamhs with a linearization in the sense that the object $Gr^W(\underline{\mathcal{U}})$ is provided with an equivalence to the Dold-Puppe of a "split pre-vmhc" (i.e. an object similar to the above but with $nGEOM$ replaced by $nHPERF$), or equivalently we include the linearizations in the above data and ask for

$$(\underline{U}_{Hod}, W, LGr^W) : S_{Hod} \to F^{DP}.nGEOM$$

and

$$(\underline{U}_{B,\mathbf{R}}, W, LGr^W) : S_{B,\mathbf{R}} \to F^{DP}.nGEOM,$$

again with an analytic equivalence

$$\zeta_{\underline{\mathcal{U}}} : (\underline{U}_{DR}, W, LGr^W)^{\mathrm{an}} \cong \zeta_S^*(\underline{U}_{B,\mathbf{R}}, W, LGr^W)^{\mathrm{an}}.$$

Suppose $\underline{\mathcal{U}}$ is a pre-vnamhs (resp. linearized pre-vnamhs) over S. As indicated above, if $s \in S$ is a point then the restriction of $\mathcal{U}$ to the various objects $\{s\}_{Hod}$ or $\{s\}_{B,\mathbf{R}}$ yields a pre-namhs (resp. linearized pre-namhs) $\underline{\mathcal{U}}(s)$.

We think that the following definition is adequate. We say that a *variation of nonabelian mixed Hodge structure* $\underline{\mathcal{U}}$ *over* S is a linearized pre-vnamhs $\underline{\mathcal{U}}$ such that for every point $s \in S$ the restriction $\underline{\mathcal{U}}(s)$ is a nonabelian mixed Hodge structure (i.e. satisfies the axioms of p. 59), and such that the family of geometric n-stacks $Tot^W(\underline{U}_{Hod}) \to S_{Hod}$, pulls back via $S \to S_{Hod}$ to a flat family over S. It is possible that it would also be necessary to apply condition **A1** "in a family" over S, but we conjecture that that is a consequence of the simple flatness hypothesis on the whole family.

Relative section stacks

Suppose $Y \to X \to B$ is a pair of morphisms of n-stacks (with at least X and B being stacks of groupoids). We can define the "sections of Y/X relative to B" which will be an n-stack over B, by the formula

$$\Gamma(X/B, Y) :=$$

$$\underline{Hom}\left(\frac{X}{B}, \frac{Y}{B}\right) \times_{\underline{Hom}(\frac{X}{B}, \frac{X}{B})} B$$

where the first morphism in the fiber product is the composition with $Y \to X$ and the second morphism is the section

$$1_{X/B} : B \to \underline{Hom}(\frac{X}{B}, \frac{X}{B})$$

corresponding to the identity in each fiber of X/B.

This has the following universal property. If $B' \to B$ is a morphism of n-stacks, let $X' := X \times_B B'$ and $Y' := Y \times_B B'$. Then a lifting

$$B' \to \Gamma(X/B, Y) \to B$$

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is the same thing as a section $X' \to Y'$ i.e. a morphism composing to the identity of X'.

We can do the same thing from a slightly different viewpoint. Suppose X and B are $n+1$-stacks (of groupoids) and suppose that Y is a cartesian family of n-stacks over X, i.e. a functor

$$Y : X \to nSTACK.$$

Then we again would like to obtain an object $\Gamma(X/B, Y)$ which, this time, should be a cartesian family of n-stacks over B. We sketch the construction here; it runs up against a limitation in [66], so a rigorous treatment would require further work. The reader may skip this discussion; it is only intended to record for future reference what types of things we would like to know in the direction of future developpment of the category-theory side of things.

Recall from [66] that for any $n+1$-stack A we have an *arrow family* which is a morphism of $n+1$-stacks

$$Arr_A : A^o \times A \to nSTACK.$$

The further information we need is that this object is *functorial in A*. To phrase this precisely, let

$$P, S : (n+1)STACK \to (n+1)STACK$$

be the functors which respectively associate to an $n+1$-stack A,

$$P(A) := A^o \times A$$

and

$$S(A) := nSTACK$$

(thus S is the constant $n+1$-stack with values $nSTACK$). The statement we need is that the arrow family construction provides a natural transformation $Arr : P \to S$.

Now suppose that $f : A \to B$ is a morphism of $n+1$-stacks and $\alpha, \beta : B \to A$ are sections of f. Then we obtain the "morphisms in A from α to β, relative to B" which is a cartesian family

$$(A/B)_{1/}(\alpha, \beta) : B \to nSTACK,$$

constructed as follows using the previous functoriality of Arr. Think of A/B as a cartesian family which we denote

$$\underline{A} : B \to (n+1)STACK,$$

and which could be heuristically denoted $b \mapsto A(b)$. Composing with the functors P (resp. S) we obtain cartesian families

$$P\underline{A} : B \to (n+1)STACK, \quad b \mapsto A(b)^o \times A(b),$$

$$S\underline{A} : B \to (n+1)STACK, \quad b \mapsto nSTACK,$$

and $Arr \circ \underline{A}$ is a natural transformation from $P\underline{A}$ to $S\underline{A}$. On the other hand $*_B$ is the cartesian family which is constant with values the one-point stack, and α and β correspond to natural transformations

$$\alpha, \beta : *_B \to \underline{A}.$$

We obtain a natural transformation $(\alpha, \beta) : *_B \to P\underline{A}$ and composing with the arrow family gives a natural transformation

$$Arr \cdot (\alpha, \beta) : *_B \to S\underline{A}.$$

Now go back from the point of view of cartesian families of $n+1$-stacks over B, to the point of view of $n+1$-stacks mapping to B: the cartesian family $*_B$ corresponds to the identity $B \to B$ and the cartesian family $S\underline{A}$ (which is constant with values $nSTACK$) corresponds to $nSTACK \times B \to B$. Our natural transformation $Arr \cdot (\alpha, \beta)$ corresponds to a morphism of $n+1$-stacks from B to $nSTACK \times B$, and composing with the first projection we obtain the desired

$$(A/B)_{1/}(\alpha, \beta) : B \to nSTACK.$$

This construction seems somewhat complicated, for an object which is "supposed to exist" by general principles; and our proposed treatment rests upon the functoriality of Arr which remains to be proven. On the other hand, it does seem to be useful for what we are planning to do below. Thus it would seem worthwhile to take a closer look at this situation in the future.

Now we get back to our proposed second construction of the relative sections stack. If $X \to B$ is a morphism of $n+1$-stacks (of groupoids) then a cartesian family $Y : X \to nSTACK$ corresponds to a section

$$B \xrightarrow{Y} A := \underline{Hom}\left(\frac{X}{B}, \frac{nSTACK \times B}{B}\right) \to B.$$

The one-point cartesian family $*_X : X \to nSTACK$ similarly corresponds to a section of A/B, and we can put

$$\Gamma(X/B, Y) := (A/B)_{1/}(*_X, Y) : B \to nSTACK$$

using the above construction.

There should, in fact, be a third construction of $\Gamma(X/B, Y)$ in the case where B is an $n + 2$-stack, X is a cartesian family over B, and Y/X is a family over B of cartesian families over X. This situation isn't formalized yet.

As usual, one would like to have a compatibility between these constructions of $\Gamma(X/B, Y)$, but we don't do that here. For the present, use the first construction which is mathematically sound. The only problem with that is that it takes place when all the stacks are n-stacks; but in the Betti case we had to use an $n + 1$-stack $X_{B,\mathbf{R}}$ for the base of a vnamhs, and technically speaking if we want to take the direct image along $X \to S$ then $S_{B,\mathbf{R}}$ should be an $n + 2$-stack. It suffices to think of everybody as $n + 2$-stacks and apply the first construction.

Extra structure

Suppose now that Y has extra structure relative to X, for example a filtration or bifiltration, eventually with linearization of one of the filtrations. Then $\Gamma(X/B, Y)$ will have the same extra structure. If (Y, W, F) is a bifiltered n-stack over X for example, then we obtain a bifiltered section stack denoted either of two ways,

$$(\Gamma(X/B, Y), W, F) = \Gamma(X/B, (Y, W, F)) : B \to F.F.nSTACK.$$

If $Y : X \to nGEOM$ is a geometric n-stack over X then the relative section stack might not necessarily be geometric. Under some circumstances it will be geometric. For example if X/B comes from a formal category which is projective and of smooth type over B, then [66] sections 8-10 give some results saying when $\underline{Hom}(X/B, Y/B)$ is geometric. The treatment of [66] concerns the case where Y comes from a family over B; one would like to improve this to treat similar cases where Y is a family over X. This corresponds to the case of local systems along the fibers of X/B.

The result of this lack of knowledge is that we can't say that the cartesian families which will appear in our proposed construction of the relative section object of a vnamhs, will be geometric or not.

Relative sections of a vnamhs

We now come to the situation which interests us. Suppose $f : X \to S$ is a projective morphism of smooth varieties. We obtain a morphism of objects $X_M \to S_M$, in other words morphisms between all of the various objects that enter into X_M and S_M, with the appropriate homotopies of compatibility between them.

We can now state the variational version of our basic Conjecture 1.

Conjecture 5 *Suppose $\underline{\mathcal{U}}$ is a (linearized) pre-vnamhs over X. Then we obtain a (linearized) pre-namhs*

$$\underline{\mathcal{G}} = \Gamma(X_M/S_M, \underline{\mathcal{U}})$$

over S. The objects going into $\underline{\mathcal{G}}$ are supposed to be the relative section stacks of the corresponding objects in $\underline{\mathcal{U}}$, for example

$$(\underline{G}_{Hod}, W) := \Gamma^{\mathrm{se}}(X_{Hod}/S_{Hod}, (\underline{U}_{Hod}, W)),$$

and

$$(\underline{G}_{B,\mathbf{R}}, W) := \Gamma(X_{B,\mathbf{R}}/S_{B,\mathbf{R}}, (\underline{U}_{B,\mathbf{R}}, W)).$$

In the first case the superscript 'se' indicates that we need to restrict to semistable sections over the Dolbeault points $S_{Dol} \subset S_{Hod}$.

If $\underline{\mathcal{U}}$ is a vnamhs over X then $\underline{\mathcal{G}}$ should be a vnamhs over S.

Part III: Computations

Now that we have made our main definitions and described the basic conjectures, we get to stating what we will actually *do* in the last part of the paper. Our work breaks into two parts:

(A) First we will construct a nonabelian mixed Hodge structure $\mathcal{V}$ whose underlying homotopy type is the complexified 2-sphere $S_{\mathbf{C}}^2$ (cf [65]). This construction is inspired by the idea that we would like to have

$$\mathcal{V} = \mathcal{MHS}(\mathbf{P}^1),$$

but we don't actually prove this latter fact. A little bit more precisely, we can obtain a morphism $\mathbf{P}_M^1 \to \mathcal{V}$ and a number of the properties required in Conjecture 3 will hold, however we don't treat the main universal property.

(B) We show the first basic Conjecture 1 for the case of $\underline{Hom}(X_M, \mathcal{V})$ for any smooth projective variety X and the namhs $\mathcal{V}$ constructed in (I) above. We hope that this proof will provide some ideas for proving Conjecture 1 in the general case.

Some morphisms between Eilenberg-MacLane pre-namhs

In the next section we will construct a namhs $\mathcal{V}$. It will be the fiber of a morphism between Eilenberg-MacLane pre-namhs's. Thus, in this section we will do a general study of morphisms between Eilenberg-MacLane pre-namhs's. This is basically the "Breen calculations" (see [7], [65], [66]) for pre-namhs's.

In this section, which prepares for the construction of $\mathcal{V}$ in the next section, we study some basic morphisms between "abelian" pre-namhs's. Recall that a *pre-mhs* is a real vector space $U_{\mathbf{R}}$ with real weight filtration W and with a "Hodge filtration" on the complexification $(\mathcal{V}_{\mathbf{C}}, F)$. By applying the construction ξ of [50] [56] [60] we obtain a pre-namhs (for $n \geq 0$)

$$\mathcal{U} = \{(U_{\mathbf{C}}, W, F), \ (U_{\mathbf{R}}, W), \ 1_{U_{\mathbf{C}}}\}.$$

Furthermore, this object is an abelian group object in $0PNAMHS$, which allows us to "deloop" it and define for any $m \leq n$ the pre-namhs denoted $K(\mathcal{U}, m)$. One way of constructing this is to let $\mathcal{U}[m]$ be the complex with $\mathcal{U}$ in degree $-m$ and zeros elsewhere; this complex is a pre-mhc, and

$$K(\mathcal{U}, m) := DP(\mathcal{U}[m]).$$

Recall (p. 53) that $\mathcal{U}$ is a *shifted mixed Hodge structure* (of *shift s*) if F and $\overline{F}$ are $k + s$-opposed on $Gr_k^W(U)$. Recall that $\mathcal{U}[m]$ will be a mixed Hodge complex (i.e. satisfies condition **MHC**) if and only if $\mathcal{U}$ is a shifted mhs of shift $-m$. More generally, $\mathcal{U}[m]$ will be an s-shifted mhc if and only if $\mathcal{U}$ is an $s - m$-shifted mhs.

We call pre-namhs's which are formed out of pre-mhs's in the above way, "Eilenberg-MacLane pre-namhs's". We need to know the "Breen calculations" for these guys, namely we would like to calculate the π_0 (or other π_i) of the space of morphisms from $K(\mathcal{U}, m)$ to $K(\mathcal{U}', m')$. Recall that the space of morphisms is denoted as

$$PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')).$$

A preliminary remark is that (because of the fact that $K(-, i)$ represents the delooping) we have

$$\pi_i PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')) =$$

$$\pi_0 PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m' - i)).$$

Thus we only need to calculate the π_0.

Theorem 7 *Suppose $m, m' \geq 2$, and suppose that m divides m' with $m' = dm$. Then for m even we have*

$$\pi_0 PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')) = Hom_{MHS}(Sym^d(\mathcal{U}), \mathcal{U}')$$

whereas for m odd we have

$$\pi_0 PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')) = Hom_{MHS}(\overset{d}{\bigwedge}(\mathcal{U}), \mathcal{U}').$$

Here Hom_{MHS} denotes the set of morphisms in the category of real mixed Hodge structures.

Proof: We write the proof in the case m even; for the case m odd say the same thing but with Sym^d replaced by $\wedge^d$.

The space of morphisms in question is a homotopy fiber product of the form

$$A \times_C B$$

where A is the space of morphisms

$$Tot^{F,W}(K(\mathcal{U}_{\mathbf{C}}, m)) \to Tot^{F,W}(K(\mathcal{U}'_{\mathbf{C}}, m'))$$

relative to $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$, where B is the space of morphisms

$$Tot^{W}(K(\mathcal{U}_{\mathbf{R}}, m)) \to Tot^{W}(K(\mathcal{U}'_{\mathbf{R}}, m'))$$

relative to $\mathcal{A}_{wt,\mathbf{R}}$, and where C is the space of morphisms

$$Tot^{W}(K(\mathcal{U}_{\mathbf{C}}, m)) \to Tot^{W}(K(\mathcal{U}'_{\mathbf{C}}, m'))$$

relative to $\mathcal{A}_{wt}$. We claim that these spaces of morphisms have all components simply connected, and have π_0 equal to the spaces of morphisms from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$, respectively: for A, morphisms of bifiltered vector spaces; for B, morphisms of real filtered vector spaces; and for C, morphisms of complex filtered vector spaces.

Before proving the claim we recall the *relative Breen calculations* in general: they say that if $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles over a base B then

$$\pi_i \underline{Hom}\left(\frac{K(\mathcal{E}/B, m)}{B}, \frac{K(\mathcal{E}'/B, m')}{B}\right) = H^{m'-i}(K(\mathcal{E}/B, m)/B, \mathcal{E}')$$

$$= \mathcal{E}' \otimes_{\mathcal{O}_B} (Sym \text{ or } \bigwedge)^{\frac{m'-i}{m}}(\mathcal{E}^*).$$

Here the symmetric product is chosen if m is even, the exterior product if m is odd; and when the exponent is fractional the answer is taken as zero.

Now getting back to the proof of the claim, note first that $Tot^{W,F}(\mathcal{U})$ is a vector bundle over $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$ and similarly for $\mathcal{U}'$; similarly $Tot^{W}(\mathcal{U}_{\mathbf{R}})$ is a real vector bundle over $\mathcal{A}_{wt,\mathbf{R}}$ and the same for $\mathcal{U}'_{\mathbf{R}}$; and finally the previous phrase also holds after complexification of everything. The Breen calculations in the relative case over $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$, over $\mathcal{A}_{wt,\mathbf{R}}$ or over $\mathcal{A}_{wt}$, then say that the relative $\underline{Hom}$ from $K(-, m)$ to $K(-', m')$ relative to the appropriate base, has relative π_0 equal to the Hom of vector bundles from the symmetric power of the first vector bundle to the second vector bundle; and the relative π_1 vanishes (this is because of the hypothesis $m \geq 2$ so the terms in the symmetric algebra are spaced out with at least one zero in between them); and the higher π_i are again vector bundles. Now the bases in question, either $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$ or $\mathcal{A}_{wt,\mathbf{R}}$ or $\mathcal{A}_{wt}$, have trivial higher cohomology with coefficients in vector bundles. Thus only the relative π_0 contributes to

the π_0 of the space of sections, and the trivial relative π_1 gives trivial π_1 of all components of the space of sections. Thus we get the simply-connectedness part of the claim, and the spaces of sections are the spaces of sections of the bundles given by the Breen calculations, i.e. the vector bundle Hom's from the symmetric powers of the bundles for $\mathcal{U}$, to the bundles for $\mathcal{U}'$. These spaces of sections are indeed the claimed spaces of morphisms from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$. This proves the claim.

Now we have identified the space of morphisms in question, as a fiber product $A \times_C B$, and we have seen that A, B and C have simply connected components. Therefore we have

$$\pi_0(A \times_C B) = \pi_0(A) \times_{\pi_0(C)} \pi_0(B).$$

In view of the above descriptions: of $\pi_0(A)$ as the space of bifiltered morphisms from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$; of $\pi_0(B)$ as the space of real filtered morphisms from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$; and of $\pi_0(C)$ as the space of complex filtered morphisms from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$, we obtain that the fibered product is exactly the space of morphisms of pre-mhs from $Sym^d(\mathcal{U})$ to $\mathcal{U}'$. This proves the theorem. |||

The following theorem could also be stated as a "conjecture" since we have only partially finished the proof; the remainder of the proof is left as an exercise.

Theorem 8 *Suppose $m, m' \geq 2$, and suppose that $m' = dm - 1$. Then for m even we have*

$$\pi_0 PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')) = Ext^1_{MHS}(Sym^d(\mathcal{U}), \mathcal{U}')$$

whereas for m odd we have

$$\pi_0 PNAMHS_{1/}(K(\mathcal{U}, m), K(\mathcal{U}', m')) = Ext^1_{MHS}(\bigwedge^d(\mathcal{U}), \mathcal{U}').$$

Here Ext^1_{MHS} denotes the Ext^1 in the category of real mixed Hodge structures. Outside of the present case and the case of the previous theorem, the answer is zero.

Sketch of Proof: Use the same proof as for Theorem 7. The only place where things change is at the line where it is stated that the relative π_0 is given by a certain formula and the relative π_1 vanishes. In the case treated by the last

sentence, both the relative π_0 and the relative π_1 of that phrase vanish, so we get that the answer is zero. In the case $m' = dm + 1$ the relative π_0 vanishes and the relative π_1 is given by the relative Breen calculations (it is the same as what occured as relative π_0 in the proof of Theorem 7). The remainder of the proof of Theorem 7 may be followed through, bearing in mind that the homotopy fiber product $A \times_C B$ is now a fiber product of connected but not simply connected spaces. This gives a certain formula. We leave as an exercise to the reader to identify the result given by this formula, with the *Ext* group in the category of real mixed Hodge structures (see Beilinson [5]) as stated in the theorem. |||

Shifting the weight filtration

Something which will be useful in the next section is the following notation. Let $D \subset \mathcal{A}$ denote the divisor defined by the origin. If need be we denote the two occurences of this as $D_{hod} \subset \mathcal{A}_{hod}$ and $D_{wt} \subset \mathcal{A}_{wt}$. Furthermore we also denote by D_{wt} the divisor

$$D_{wt} \times \mathcal{A}_{hod} \subset \mathcal{A}_{wt} \times \mathcal{A}_{hod}.$$

If $\mathcal{U} = (U, W)$ is a filtered vector bundle then we denote by $\mathcal{U}(-D)$ the filtered bundle whose total bundle of the weight filtration is

$$Tot^W(\mathcal{U}(-D)) := Tot^W(U) \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{A}}(-D).$$

This corresponds to renumbering or "shifting" the filtration W, namely

$$W_{m+1}(\mathcal{U}(-D)) = W_m(\mathcal{U}).$$

To see that this is the way the renumbering works, note that $u \in W_m(\mathcal{U})$ if and only if $t^m u$ is a section of $T^W(\mathcal{U})$ (see p. 16). We have the inclusion of locally free sheaves over $\mathbf{A}^1$

$$T^W(\mathcal{U}(-D)) \hookrightarrow T^W(\mathcal{U}),$$

and $t^m u$ is a section of $T^W(\mathcal{U})$ if and only if $t^{m+1}u$ is a section of $T^W(\mathcal{U}(-D))$, in other words

$$u \in W_m(\mathcal{U}) \Leftrightarrow u \in W_{m+1}(\mathcal{U}(-D)).$$

In the bifiltered situation we adopt a similar notation which gives rise to the same notation for pre-mhs's: if $\mathcal{U}$ is a pre-mhs then we obtain a new

pre-mhs $\mathcal{U}(-D_{wt})$ where the weight filtration is shifted by one. Note that there is a canonical inclusion

$$\mathcal{U}(-D_{wt}) \to \mathcal{U}$$

which induces the zero map on the associated-graded Gr^W, and in fact $\mathcal{U}(-D_{wt})$ can be characterized as the universal such object: any morphism of pre-mhs $\mathcal{U}' \to \mathcal{U}$ which induces zero on Gr^W, factors uniquely through the subobject $\mathcal{U}(-D_{wt})$.

Construction of a namhs $\mathcal{V}$ of homotopy type $S_{\mathbf{C}}^2$

We would like to construct our $\mathcal{V}$ to have underlying homotopy type V_B equal to the complexified 2-sphere $S_{\mathbf{C}}^2$ (cf [65]). This means that we want $\mathcal{V}$ to fit into a Postnikov fibration sequence

$$K(\mathcal{U}', 3) \to \mathcal{V} \to K(\mathcal{U}, 2).$$

The twisting class should be nontrivial (as soon as this class is nontrivial the underlying homotopy type will be $S_{\mathbf{C}}^2$).

In fact, we would like $\mathcal{V}$ to model the homotopy type of $\mathbf{P}^1$ (since we know, *a priori*, that this should have a "mixed Hodge structure").

We have to choose the shifted mixed Hodge structures $\mathcal{U}$ and $\mathcal{U}'$ appropriately; we look at how to do this in a moment. First note that the above sequence is obtained by specifying the *classifying morphism*

$$Q : K(\mathcal{U}, 2) \to K(\mathcal{U}', 4).$$

To be precise, $\mathcal{V}$ is the fiber of the map Q, thus it is the homotopy fiber product

$$\mathcal{V} = K(\mathcal{U}, 2) \times_{K(\mathcal{U}', 4)} *.$$

By the Breen-type calculations of the previous section, this is equivalent to specifying a morphism of pre-mhs

$$Sym^2(\mathcal{U}) \to \mathcal{U}'.$$

Another thing to note is that we would like the resulting $\mathcal{V}$ to support a linearization; this means that the associated-graded of the morphism Q for the weight filtration, should be zero. Another thing to notice is that we

would like the base $K(\mathcal{U}, 2)$ and the fiber $K(\mathcal{U}', 3)$ to satisfy condition **MHC**. Finally, for our example we would like $\mathcal{V}$ to model the homotopy type of $\mathbf{P}^2$. Thus $\mathcal{U}$ and $\mathcal{U}'$ should be one-dimensional.

In our case of rank one structures, the real structure and the Hodge filtration determine a pure Hodge structure of some type (p, p); on the other hand, the weight filtration has only one nontrivial step, at weight w. Thus the pre-mhs's $\mathcal{U}$ and $\mathcal{U}'$ are determined respectively by specifying p and w (resp. p' and w').

The condition that the morphism Q should be zero on the associated-graded, is equivalent to saying that Q factors through a morphism

$$Q' : Sym^2(\mathcal{U}) \to \mathcal{U}'(-D_{wt})$$

(see the end of the previous section p. 110 for the notation D_{wt}).

We would like to get a $\mathcal{V}$ which models the homotopy type of $\mathbf{P}^1$. During our original research on this question, we actually started out by looking at the homotopy type of $\mathbf{P}^1$ and then worked backwards from there to deduce what conditions needed to be required of a nonabelian mixed Hodge structure. In particular, one notices from the $\mathbf{P}^1$ example: the fact that one needs to include the "homotopy weight" in the picture (although this was known from another context of Hodge III [14]); and the condition that the Whitehead product induce zero on the associated-graded of the weight filtration, which suggested the general definition that the associated-graded of the weight filtration should have a structure of a "spectrum" or equivalently a perfect complex.

To get back to the arithmetic at hand, we need to specify p, w, p' and w'. Since we would like to have $\mathcal{U}$ be $\pi_2(\mathbf{P}^1)$ and $\mathcal{U}'$ be $\pi_3(\mathbf{P}^1)$, we take $p = -1$ and $p' = -2$. This determines the weights, namely $w = 0$ and $w' = -1$.

This works out correctly for the condition **MHC**: note that $LGr^W(\mathcal{V})$ will be the product of $K(\mathcal{U}, 2)$ and $K(\mathcal{U}', 3)$. We have (according to the choices of w and w')

$$H^{-i}(LGr_k^W(\mathcal{V})) = \begin{array}{ll} \mathcal{U}, & k = w = 0, \ i = 2; \\ \mathcal{U}', & k = w' = -1, \ i = 3; \\ 0 \ \text{otherwise} \end{array} .$$

The purity condition **MHC** (cf p. 55) says that on $H^{-i}(LGr_k^W(\mathcal{V}))$, the Hodge filtration F and its complex-conjugate $\overline{F}$ (taken with respect to the

real structure), are m-opposed with

$$m = k - i.$$

Note that according to our choice of p and p', we obtain $m = 2p = -2$ for $\mathcal{U}$ (which works out as $-2 = m = k - i = 0 - 2$) and $m = 2p' = -4$ for $\mathcal{U}'$ (which works out as $-4 = m = k - i = -1 - 3$). Thus, with our above choices of p, w, p', w' we do indeed obtain the condition **MHC** for $\mathcal{V}$.

Theorem 9 *The extension $\mathcal{V}$ constructed as the fiber of the morphism Q (with p, w, p' and w, chosen as above), is a nonabelian mixed Hodge structure.*

Proof: Note that all of the total spaces of filtrations involved in $\mathcal{V}$ are smooth, so the flatness and annihilator conditions are trivial. Our choice of p, w, p' and w' gives that $K(\mathcal{U}, 2)$ and $K(\mathcal{U}', 3)$ are both nonabelian mixed Hodge structures, i.e. they satisfy the condition **MHC**, as calculated immediately above. Note that they also satisfy the strictness condition **Str**. The linearized associated graded $LGr^W(\mathcal{V})$ is a direct sum of those for $K(\mathcal{U}, 2)$ and for $K(\mathcal{U}', 3)$ (because the morphism Q is zero on the associated graded) so it satisfies condition **MHC**. |||

Remark: Note that $K(\mathcal{U}', 4)$ does *not* satisfy the **MHC** condition (since it is translated by one homotopical degree). In particular, we can't say that Q is a morphism of namhs. On the other hand, $K(\mathcal{U}'(-D_{wt}), 4)$ is a namhs, i.e. it does satisfy the condition **MHC**, so the morphism Q' is a morphism of namhs which makes it "strict" in a certain sense. This phenomenon plays an important role in what goes on at the end of the proof in the next section.

The namhs on cohomology of a smooth projective variety with coefficients in $\mathcal{V}$

Suppose now that X is a smooth projective variety. We consider the situation of Conjecture 1, for the particular namhs $\mathcal{V}$ constructed above (Theorem 9).

Theorem 10 *If X is a smooth projective variety and $\mathcal{V}$ is the nonabelian mixed Hodge structure constructed in Theorem 9, then the linearized pre-namhs*

$$\mathcal{H} := \underline{Hom}(X_M, \mathcal{V})$$

constructed in Theorem 4, is a nonabelian mixed Hodge structure.

The proof occupies the remainder of the paper. Start by writing

$$\mathcal{H} = \underline{Hom}(X_M, \mathcal{V}) =$$

$$\underline{Hom}(X_M, K(\mathcal{U}, 2)) \times_{\underline{Hom}(X_M, K(\mathcal{U}', 4))} \underline{Hom}(X_M, *).$$

We have to analyse this fiber product. Note that $\underline{Hom}(X_M, *) = *$.
    Put

$$\mathcal{E} := \underline{Hom}(X_M, K(\mathcal{U}, 2))$$

and

$$\mathcal{F} := \underline{Hom}(X_M, K(\mathcal{U}', 4)).$$

Also put

$$\mathcal{F}' := \underline{Hom}(X_M, K(\mathcal{U}'(-D_{wt}), 4)),$$

where D_{wt} was the divisor $\mathcal{A}_{hod} \times [0]_{wt} \subset \mathcal{A}_{hod} \times \mathcal{A}_{wt}$. These are all pre-mhc's,
and we have a morphism $i : \mathcal{F}' \to \mathcal{F}$ inducing zero on the associated-graded of
the weight filtration. We have constructed above a morphism (of underlying
pre-namhs's) Q from $K(\mathcal{U}, 2)$ to $K(\mathcal{U}', 4)$, which in turn yields

$$Q_{\mathcal{H}} : DP\mathcal{E} \to DP\mathcal{F}$$

factoring through $Q'_{\mathcal{H}} : DP\mathcal{E} \to DP\mathcal{F}'$. The fact that i induces zero on
the Gr^W implies that $Q_{\mathcal{H}}$ extends to a morphism of linearized pre-namhs's.
On the other hand the zero-section is a morphism (again, of linearized pre-
namhs's)

$$z : * \to DP\mathcal{F}.$$

We have

$$\mathcal{H} = \mathcal{E} \times_{\mathcal{F}} *,$$

in other words $\mathcal{H}$ is the fiber of $Q_{\mathcal{H}}$ over the zero-section.

Lemma 26 *In the above situation, $\mathcal{E}$ is a mixed Hodge complex and $\mathcal{F}$ is a
1-shifted mhc.*

Proof: This is a trivial case of [14], see also Theorem 5. The only difficulty
is that $\mathcal{E}$ and $\mathcal{F}$ are truncations of the mixed Hodge complexes given by [14].
Thus, *a priori* they are only truncated pre-mhc's. However, in the present
trivial case (X being smooth projective), the weight filtration is strict too
(its spectral sequence degenerates at E_1) which means that the cohomology

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objects are bundles over $\mathcal{A}_{wt}$; thus truncating yields again a complex whose cohomology objects are bundles, i.e. it is a perfect complex.

For the sign of the shift for $\mathcal{F}$, note that $K(\mathcal{U}', 4)$ has Hodge degree $h = -4$, weight $w = -1$, and homotopical degree $i = 4$. Now $h = (w - i) + 1$ which means that $K(\mathcal{U}', 4)$ is 1-shifted; thus $\mathcal{F}$ is also 1-shifted. |||

Corollary 27 *The pre-namhs $\mathcal{H}$ satisfies the strictness condition* **Str** *as well as the condition* **MHC**.

Proof: This was pointed out in Corollary 22; for the record we recall how this goes in this particular case. The connecting morphism Q induces zero on the associated-graded Gr^W, so

$$LGr^W(\mathcal{H}) = LGr^W(\mathcal{E}) \times \Omega LGr^W(\mathcal{F}).$$

Here Ω is the loop-space, which on the level of complexes means shifting to the right by one and truncating at 0. As in the lemma, the mixed Hodge complex $\mathcal{F}$ has a strict weight filtration in our easy situation, so taking the truncation of the shift again yields a pre-mhc. Recall that $\mathcal{F}$ was a 1-shifted mhc; taking the loop-space undoes this shift so $\Omega LGr^W(\mathcal{F})$ is a mixed Hodge complex. Thus $LGr^W(\mathcal{H})$ is a mixed Hodge complex (which includes the strictness condition). |||

Some general theory

In order to analyze the annihilator ideals in the fiber product $\mathcal{E} \times_{\mathcal{F}} *$ we discuss this general type of situation.

Lemma 28 *Suppose $f : X \to Y$ is a morphism of geometric n-stacks which induces an isomorphism on the level of π_0, and is surjective on π_1. Then f is smooth.*

Proof: A morphism between geometric n-stacks is smooth if and only if it is formally smooth (this remark was made e.g. in [66] §7.3), thus it suffices to prove that f is formally smooth. Suppose $A \subset B$ are artinian local schemes. The problem is to show that the map

$$X(B) \to X(A) \times_{Y(A)} Y(B)$$

is surjective on π_0. Note that the component maps $X(B) \to Y(B)$ and $X(A) \to Y(A)$ are isomorphisms on π_0 and surjections on π_1. These mean that the fibers of these morphisms are connected. We obtain from the statement for A that the fibers of the map

$$ X(A) \times_{Y(A)} Y(B) \to Y(B) $$

are connected, in particular this map induces an isomorphism on π_0; but then from the statement for B we obtain that the map

$$ \pi_0(X(B)) \to \pi_0(X(A) \times_{Y(A)} Y(B)) = \pi_0(Y(B)) $$

is an isomorphism. This yields the required lifting property. |||

Corollary 29 *Suppose that X is a geometric n-stack and suppose that $\pi_0(X)$ is represented by an algebraic space (i.e. it is a geometric 0-stack). Then the morphism $X \to \pi_0(X)$ is smooth.*

Proof: Indeed, it clearly satisfies the hypotheses of the previous lemma. |||

The same argument gives furthermore

Corollary 30 *Suppose that X is a geometric n-stack and suppose that there is $k \leq n$ such that $\tau_{\leq k}(X)$ is a geometric k-stack. Then the morphism $X \to \tau_{\leq k}(X)$ is smooth.*

Proof: Again, it satisfies the hypotheses of the previous lemma. |||

The above smooth morphisms are useful for the following reason.

Lemma 31 *Suppose $X \xrightarrow{g} Y \to \mathbf{A}^1$ is a morphism of geometric n-stacks over $\mathbf{A}^1$. If g is smooth, then the annihilator ideals for X are the pullbacks of the annihilator ideals for Y.*

Proof: This follows from the definition of annihilator ideals, since a smooth chart for X gives by composing with g a piece of a smooth chart for Y (if g wasn't surjective it might be necessary to add on some other pieces to the chart). |||

Corollary 32 *Suppose B is a scheme, and suppose X and Y are geometric n-stacks over B. Suppose that $U = \pi_0(X)$ and $V = \pi_0(Y)$ are represented by schemes. Fix a section $b : B \to Y$ and let F denote the fiber of $X \to Y$ over b, in other words,*

$$F = X \times_Y B.$$

Let G be the fiber of $U \to V$ over the projection pb (here p denotes the projection from Y to V). Then the morphism

$$F \to G$$

is smooth.

Proof: We can base change everything by the map $B \to V$, in particular we can suppose that $V = B$. Note then that $G = U$. With this hypothesis, our section $b : B \to Y$ induces an isomorphism on π_0, so by Lemma 28 it is smooth. This implies that

$$F = X \times_Y B \to X$$

is smooth. On the other hand, the morphism $X \to U$ is smooth, again by Lemma 28. Thus the composition $F \to U = G$ is smooth. |||

We apply this to the case where the base is $B = \mathbf{A}^1$. We conclude that the annihilator ideals for the fiber F are the pullbacks of the annihilator ideals for the fiber G of $\pi_0(X) \to \pi_0(Y)$ (under the hypothesis that these π_0 are schemes). The same remark holds for $B = \mathbf{A}^1 \times \mathbf{A}^1$ where we calculate the annihilator ideals with respect to one of the variables. Similarly, flatness of the fiber with respect to one of the variables is equivalent to flatness of the fiber with respect to the fiber of the map on the π_0.

To conclude, we find that in order to check the annihilator conditions and flatness conditions for the fiber of a morphism of linearized pre-namhs, if the π_0 are schemes (say for the total space of the Hodge and weight filtrations over $\mathbf{A}^1 \times \mathbf{A}^1$) then it suffices to check these conditions for the fiber of the induced morphism on π_0.

Lemma 33 *Suppose $\mathcal{B}$ and $\mathcal{C}$ are linearized pre-namhs's, and suppose that $f : \mathcal{B} \to \mathcal{C}$ is a smooth morphism. Suppose that both $\mathcal{B}$ and $\mathcal{C}$ satisfy the "mixed Hodge complex" condition* **MHC** *and strictness* **Str**. *Then $\mathcal{B}$ is a namhs if and only if $\mathcal{C}$ is a namhs.*

Proof: The point is to say that since f is smooth, the annihilator ideals of $\mathcal{B}$ are the pullbacks of those for $\mathcal{C}$ (and also flatness of the Hodge filtration is equivalent on $\mathcal{B}$ or $\mathcal{C}$); thus the remaining conditions **A1**, **A2**, **Fl** and **A3** for $\mathcal{B}$ and for $\mathcal{C}$, are equivalent. |||

Continuation of the proof of Theorem 10

We analyze our fiber $\mathcal{H}$ using the general theory. Note first of all,

Lemma 34 *In the above situation, the relative zeroth-homologies $H^0(\mathcal{E})$, $H^0(\mathcal{F})$ and $H^0(\mathcal{F}')$ are themselves pre-mhs's, in particular*

$$Tot^{W,F} H^0(\mathcal{E})_{DR} = H^0(Tot^{W,F}(E_{DR})/\mathcal{A}_{hod} \times \mathcal{A}_{wt})$$

is a vector bundle over $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$, and the same for $\mathcal{F}$ and $\mathcal{F}'$.

Proof: This strictness comes from the fact that, exceptionally in this case, the spectral sequences for the weight filtration degenerate at E_1 rather than E_2, as in Lemma 26. |||

Note that

$$\pi_0(DP\mathcal{E}) = H^0(\mathcal{E}), \quad \pi_0(DP\mathcal{F}) = H^0(\mathcal{F}).$$

The morphism Q is not linear so we represent it as a morphism between π_0.

Corollary 35 *We get a morphism of linearized pre-namhs's which are schemes*
$$Q_{\mathcal{H},0} : \pi_0(DP\mathcal{E}) \to \pi_0(DP\mathcal{F}).$$
Let $\mathcal{K}$ be the fiber of this morphism over the zero section of $\pi_0(DP\mathcal{F})$. Then there is a smooth morphism of linearized pre-namhs's

$$\mathcal{H} \to \mathcal{K}.$$

Proof: From above. |||

Corollary 36 *In our situation of Corollary 35, the linearized pre-namhs $\mathcal{H}$ is a nonabelian mixed Hodge structure if and only if $\mathcal{K}$ is.*

Thus to finish the proof of Theorem 10 it suffices to show that $\mathcal{K}$ is a namhs.

We now analyze the situation a bit further. Recall that $\mathcal{U}$ is a pure Hodge structure of Hodge type $(-1,-1)$ with a one-step weight filtration placed at weight 0, and $\mathcal{U}'$ is a pure Hodge structure of type $(-2,-2)$ with weight filtration placed at weight -1.

We have
$$H^0(\mathcal{E}) = H^2(X, \mathbf{C}) \otimes \mathbf{C}^{-1,-1}\{0\}$$
and
$$H^0(\mathcal{F}) = H^4(X, \mathbf{C}) \otimes \mathbf{C}^{-2,-2}\{-1\}.$$

The brackets represent the weight of the pre-mhs in question. Note that $H^0(\mathcal{E})$ is a 0-shifted mhs, pure of weight zero. On the other hand, $H^0(\mathcal{F})$ is a 1-shifted mhs, where the Hodge filtration is pure of weight zero but the weight filtration is shifted and located at -1.

The quadratic morphism $Q_{\mathcal{H},0}$ is just cup-product, composed with the inclusion $\mathbf{C}\{0\} \to \mathbf{C}\{-1\}$.

Recall that $\mathcal{F}' = \mathcal{F}(-D_{wt})$. The morphism $Q_{\mathcal{H},0}$ factors through a morphism
$$Q'_{\mathcal{H},0} : \pi_0(DP\mathcal{E}) \to \pi_0(DP\mathcal{F}')$$
(this latter being just cup-product, without composition with the inclusion $\mathbf{C}\{0\} \to \mathbf{C}\{-1\}$). Let $\mathcal{K}'$ denote the zero-subscheme of $Q'_{\mathcal{H},0}$.

Note in the above terms that
$$\pi_0(DP\mathcal{F}') = H^0(\mathcal{F}') = H^4(X, \mathbf{C}) \otimes \mathbf{C}^{-2,-2}\{0\}.$$

It is a 0-shifted mixed Hodge structure which is in fact pure of weight zero (see the renumbering of page 110). Recall that the same was true of $H^0(\mathcal{E})$. In particular $Q'_{\mathcal{H},0}$ is a quadratic morphism of mixed Hodge structures. We get back to $Q_{\mathcal{H},0}$ by multiplying $Q'_{\mathcal{H},0}$ by the coordinate t defining the divisor $D_{wt} \subset \mathcal{A}_{wt}$.

In the above particular case, the total spaces
$$Tot^{F,W}H^0(\mathcal{E}) \quad \text{and} \quad Tot^{F,W}H^0(\mathcal{F}')$$
are bundles over $\mathcal{A}_{hod} \times \mathcal{A}_{wt}$. The weight filtrations are one-step and in degree zero, so these bundles are actually pulled back from $\mathcal{A}_{hod}$. Furthermore,

the Hodge filtrations have splittings by the Hodge decomposition, and the cup-product is compatible with the splitting. Thus on $\mathbf{A}^1_{hod}$ the bundles in question become trivial bundles, and the quadratic form Q' is also trivial i.e. it is a product. Therefore the pullback of $\mathcal{K}'$ to $\mathbf{A}^1_{hod} \times \mathbf{A}^1_{wt}$ has a structure of product

$$\mathcal{K}' \times_{\mathcal{A}_{hod} \times \mathcal{A}_{wt}} \mathbf{A}^1_{hod} \times \mathbf{A}^1_{wt}$$

$$\cong \mathbf{A}^1_{hod} \times \mathbf{A}^1_{wt} \times C$$

where C is the quadratic cone in $H^2(X, \mathbf{C})$, kernel of the cup-product map to $H^4(X, \mathbf{C})$. This product decomposition is compatible with the inclusion into $Tot^{F,W} H^0(\mathcal{E})$ (or rather the restriction $T(\mathcal{E})$ of this latter over $\mathbf{A}^1_{hod} \times \mathbf{A}^1_{wt}$). Finally, $\mathcal{K}$ is the subscheme of $T(\mathcal{E})$ defined by the equation of $\mathcal{K}'$, multiplied by t. Thus $\mathcal{K}$ is the scheme-theoretic union of $\mathcal{K}'$ and the inverse image of the divisor D_{wt}. In particular it is trivializable in the $\mathbf{A}^1_{hod}$-direction, so F is flat. Also, the structure as union of something flat, with the inverse image of the reduced divisor D_{wt}, means that the annihilator ideal condition **A1** is satisfied. The annihilator ideal $\ker(t)$ is just the ideal defining $\mathcal{K}'$, and $Ann^W(t; \mathcal{H})$ is the restriction of this over $D_{wt} = [0]_{wt} \times \mathcal{A}_{hod}$. Condition **A2** is satisfied because $\mathcal{K}'$ is nonempty. Finally, the cup-product morphism is a morphism of mixed Hodge structures, so the ideal defined by it is a sub-mixed Hodge structure in the coordinate ring of $H^2(X, \mathbf{C})$. This is condition **A3**. ‖‖

It would seem to be a reasonable project to try to replicate the above types of computations, for objects $\mathcal{V}$ supposed to represent the homotopy types of projective spaces $\mathbf{P}^n$ or Grassmanian varieties. These types of explicit computations would present an interest of their own, independant of any eventual general proof of Conjecture 1.

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